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A COLLECTION *of* PAPERS

in memory of

Sir William Rowan Hamilton

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The year 1943 is the one hundredth anniversary of the epoch-making discovery of Quaternions by Sir William Rowan Hamilton. If the world were at peace this anniversary would have become an occasion for celebrations by scientists all over the world and for the publication of monographs and papers dealing with the evaluation of the various contributions made by the great Irish mathematician to Mathematics and to Science, and with their influence on modern thought.

But the world is engaged in a life-and-death struggle and cannot be distracted by anniversaries of scientists or of scientific discoveries. Only a small number of institutions in this country and abroad have paused to pay tribute to Hamilton.

The editors of Scripta Mathematica are glad to have this volume dedicated to the memory of Hamilton. It contains papers dealing with a number of important phases of Hamilton's life and scientific activity, and it is hoped that these will be found to be a genuine contribution to the literature on Hamilton.

The editors acknowledge with thanks the cooperation of the Royal Irish Academy of Sciences and of the Honorable Leo T. McCauley, Consul General of Ireland in New York, in securing for Scripta Mathematica the portrait of Hamilton used as a frontispiece and of the related pictorial items.

Sir William Rowan Hamilton
with one of his sons
circa 1845

SIR WILLIAM ROWAN HAMILTON

BY DAVID EUGENE SMITH*

Among the "infant prodigies" in the field of mathematics, and particularly among the small number who later developed into scholars of high rank, Sir William Rowan Hamilton stands out conspicuously. Born in Dublin, his mother was Scotch and his father was of Irish birth but with remote Scottish ancestry, as the family name suggests. Whatever his nationality, however, he was proud of his Irish blood and he passed his life in the land of his birth.

Beginning his school work under the care of an uncle who was a clergyman possessed of a high degree of scholarship, at the age of three he read English fluently and was well along in elementary arithmetic. When he reached his fifth birthday he was so far advanced as to read Latin, Greek, and Hebrew—the three languages generally required of the learned class at that time. He was not allowed to neglect the best of the English literature, however, and was able to recite long passages from such writers as Dryden and Milton. When he was eight years old he could converse in Latin and two years later he was studying Arabic and Sanskrit. When he was twelve he came in contact with an American boy, also of the "infant prodigy" type and remarkably gifted in mental arithmetic, as it was called. This boy, Zerah Colburn, was able to perform mentally long operations with numbers, and his ability in this field stimulated the young Hamilton to cultivate the same power. So skilful did he become that all through his life he took pleasure in performing long operations with numbers, including the finding of square and cube roots.

* Reprinted from his *Portraits of Eminent Mathematicians, with Brief Biographical Sketches*, Portfolio II, published in 1938 by SCRIPTA MATHEMATICA.

It frequently happens that children who show unusual powers in mental calculations are content to stop there and therefore fail to make any noteworthy progress in mathematics, but this was not the case with Hamilton. He continued his study of languages, and when he reached the age of thirteen he began the study of algebra, taking for his textbook a work written nearly a century earlier, by Clairaut, one of the leading French mathematicians. It is a curious circumstance that this same Clairaut was himself an "infant prodigy," for at the age of ten he read two very advanced works on the calculus. When he was only thirteen he presented a paper on geometry before the Académie des Sciences in Paris, and was admitted to membership in this famous society when he was eighteen. It would seem that the style of the young French scholar was such as to capture the attention of the young Irish student who was to astonish the world many years later.

As to Hamilton, we find him at the age of fourteen, writing a letter in the Persian language, addressed to the Persian ambassador who was making a visit to Ireland. Even to read Persian at such an early age would be an achievement that would attract attention, but to compose a letter in that language and particularly to write it in the script of a country which he had never visited, was more than simply remarkable.

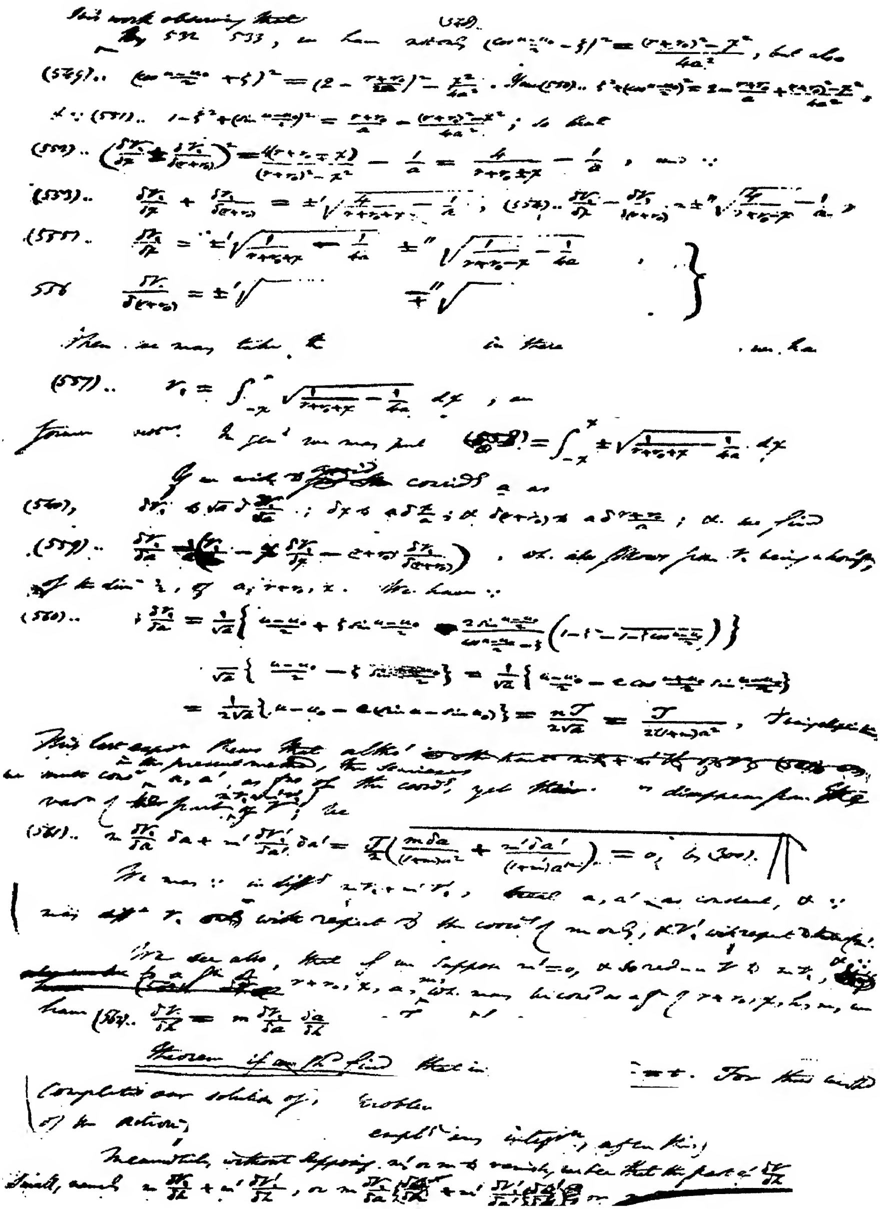
Two years later found Hamilton studying the differential calculus from a French textbook, but this was not his first experience with mathematics of importance, for he had read Newton's Latin work on algebra, the *Arithmetica Universalis*, when he was only twelve years old, and with this as a beginning he advanced to Newton's famous *Philosophiae Naturalis Principia Mathematica*, a work which is by no means easy reading for rather advanced college students of our day, even in the English edition.

So brilliant a scholar could not fail to attract the attention of the learned world. It was his reading of the *Mécanique*

Céleste at the age of seventeen, a work by the famous French scholar Laplace, that brought him suddenly to the front rank of the mathematical world. In this work he detected an error and communicated the fact to Brinkley, the Astronomer Royal of Ireland, with his correction. After this he was looked upon not merely as Ireland's greatest scholar in his field, but as one of international standing—and this when only seventeen. At the age of twenty-two he was made professor of astronomy in Dublin, following Brinkley, and the rest of his life was devoted to the study of mathematics and astronomy at the observatory near Dublin. He was knighted in 1835, at the early age of thirty.

Hamilton is known in the mathematical world chiefly for his work on quaternions, a word derived from the Latin *quatuor* (four), because it depends on four geometric elements. It may be called an algebra of vectors. It was at once recognized as of great importance in the study of physics, but the later subject of vector analysis has largely supplanted it in all its former applications.

Hamilton's *Lectures on Quaternions* appeared in 1853, and his *Elements of Quaternions* in 1866 (posthumously). He was the first Foreign Associate of the National Academy of the United States, one of numerous evidences of the world's appreciation of his ability as a mathematician, astronomer, and physicist.



A Page from one of Hamilton's Notebooks
Courtesy of the Royal Irish Academy

THE LIFE AND EARLY WORK OF SIR WILLIAM ROWAN HAMILTON

By J. L. SYNGE

Introduction

WHO was Hamilton? An Irishman, who lived in the first half of the nineteenth century, and was regarded by his contemporaries as one of the greatest mathematicians of the period. What did he leave to posterity? Omitting lesser contributions, he left two things: first, the principal or characteristic function, and, secondly, quaternions. Since the accompanying article by Professor MacDuffee deals with quaternions, I shall make only one remark here. Hamilton (and doubtless many of his contemporaries) thought that quaternions would become a universal notation in geometry and applied mathematics. They have not done so, and it seems unlikely that they ever will. Their rôle has been taken over by vectors, tensors, and matrices. The quaternion must be regarded as a stimulus to invention rather than as a permanent recognized notation.

The purpose of this article is to give a brief account of Hamilton's life, and of his other great idea—the principal or characteristic function. Since, to mathematicians, the life of a mathematician is of interest only as a background to his work, I shall leave the biographical details to the end.

The Principal or Characteristic Function

Mathematicians are human beings. They pursue mathematics under the influence of a subtle urge, which varies from man to man and from age to age, but preserves throughout a measure of consistency. Thus every modern mathematician will find in himself some impulses identical with the impulses of the mathematician of former days, but he must be prepared to find differences.

A bald recital of formulae which Hamilton developed would not fulfill my purpose. We must get a broad perspective of the age in which he

lived (1805–1865). His life overlapped the lives of Lagrange (1736–1813), Laplace (1749–1827), and Gauss (1777–1855). This was the heroic age of mathematics, before the elaboration of modern experimental technique. The mathematician was, and knew that he was, the mightiest figure in science. His work interpreted man's relationship to nature, and it is not too much to say that mathematics in the heroic age was a religion. For better or worse, that spirit seems to be gone. What modern mathematician dreams that his research is going to make one jot of difference to the philosophy of life of the human race? Galileo did this, and Newton did this, but the age is past. It was passing even at the beginning of the nineteenth century, but neither the public nor the mathematicians knew it. There is a lag in the intellectual affairs of men, which enables a generation to live partly in the mentality of a previous generation.

When Hamilton's young mind crystallized into a pattern, it was the pattern of the late eighteenth century. It was not his ambition to polish the corners of structures built by other men. Each great figure had a monument enshrining a great idea—a great idea connected not with the mere creation of a mathematical technique, but a great idea which revealed nature to man through the exercise of transcendent mathematical genius. Newton had the theory of gravitation and planetary motions, Lagrange had his dynamical equations, Laplace had the theory of potential. What monument would Hamilton create to make his memory imperishable?

Hamilton began with the theory of light at the age of seventeen, and worked at this theory until he was thirty, returning to it intermittently during the next ten years. In the course of this work, he realized that optics and dynamics are essentially a single mathematical subject, and so his activities covered dynamics as well as optics. But we would miss the point entirely if we were merely to assert that Hamilton made outstanding contributions to optics and dynamics. That would not reveal the motivating spirit of the heroic age. All his work in this field centered round one idea—an idea conceived in such a spirit of generality as to place Hamilton in the same category as Descartes, Lagrange, or Laplace. For to each of these men there can be pinned a label with a single legend. Descartes characterized or described a curve by writing down a single equation. Lagrange characterized or described a dynamical system by writing down a single energy function. Laplace characterized or described a gravitational field by writing down a single potential function. Each of these concepts has the simplicity of greatness. It formed a frame into which countless lesser mathematicians were to fit a mosaic of detail.

If I say that Hamilton followed in the footsteps of these men, I am not drawing on my imagination. He stated so himself. His plan was to characterize or describe any optical or dynamical system by means of a single characteristic or principal function. This is not to be confused with Lagrange's idea. Hamilton's principal function is a more subtle concept than Lagrange's energy function.

The distinctions which came to Hamilton during the first forty years of his life testify that contemporary mathematicians realized the greatness of his central idea. It is, however, safe to say that very, very few of his contemporaries appreciated his idea to the full. They picked on outlying developments which were intrinsically important but which did not involve the *one* central idea—the characteristic or principal function.

Perhaps it would make things clearer for the modern mathematician if I were to leave out the physical language and use the language of pure mathematics. To Hamilton, optics and dynamics were merely two aspects of the calculus of variations. He was not interested in experiments. To him, optics was the investigation of the mathematical properties of curves giving stationary values to an integral of the type

$$V = \int v\left(x, y, z, \frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du}\right)du.$$

Hamilton was, in fact, a great contributor—probably the greatest single contributor of all time—to the calculus of variations.

Consider two points, A' with coordinates x', y', z', and A with coordinates x, y, z. Consider all possible curves connecting A' and A. To each curve there corresponds a value of the above integral. Now compare these values. Is there one curve that gives a smaller value to the integral than the others? It would seem that there must be, but it would be a rash conclusion. Suffice it today that "in general" there is a curve giving a smallest value to the integral. Then such a curve is a *ray* in optics. In the language of the calculus of variations, it is an extremal. (These words cover also curves for which the integral has a "stationary" value.)

Now here is Hamilton's great central idea. *Regard the minimum (or stationary) value of the integral as a function of the six coordinates x', y', z', x, y, z, of A' and A.* This function is Hamilton's characteristic or principal function. There is nothing hard to understand about this. What is hard to follow is Hamilton's plan to develop all the properties of the extremals from this characteristic function. In fact,

it is so hard that few mathematicians in the ensuing century have given serious consideration to his plan.

To explain this neglect involves a study of mathematical psychology. I have said that Hamilton belonged to the heroic age of mathematics. He regarded his work as a form of art, as indeed do many mathematicians today. But there is a difference. The modern mathematician weaves an intricate pattern of microscopic precision. To him, a false statement—an exception to a general statement—is an unforgivable sin. The heroic mathematician, on the other hand, painted with broad splashes of color, with a grand contempt for singular cases until they could no longer be avoided. He aimed at creating a picture in which his artistry was revealed by the originality of the conception and the symmetry of the composition, rather than by a faultless precision in detail.

I would, however, be creating an entirely false impression of Hamilton if I were to imply that he lived in the clouds and spurned details. But the details with which he busied himself were not the details that the modern mathematician considers important. Few modern mathematicians like to work with a function without a careful statement regarding its general properties (continuity and existence of derivatives), or without assurance of existence under stated conditions. Hamilton, in the spirit of the heroic age, goes right ahead. That is the feature which doubtless repels most modern mathematicians from a more careful study of his work. It explains, in large measure, the eclipse of Hamilton.

It seems, however, a pity to allow great ideas to fall into discard merely because they are archaically expressed. The eclipse of Hamilton is greater than it should be.

Let us see what has happened to some of his work. In the earliest applications of the characteristic function (1824), Hamilton gave the first general treatment of rectilinear congruences. In 1860, Kummer, with due acknowledgment of Hamilton's long priority, generalized Hamilton's work. Now Hamilton's work is forgotten, except for his name in a formula, while Kummer's is remembered.[1]

Hamilton developed his idea of a characteristic function first in optics. The general impression seems to have been: "Very beautiful, but useless." Perhaps this was because readers did not trouble to dig deeply enough into Hamilton's work to discover that he had several types of characteristic function, and the criticism aimed at the first was much less applicable to the last. However, in 1895, Bruns[2]

[1] Cf. L. P. Eisenhart, *Differential Geometry* (Boston, 1909), p. 392.

[2] H. Bruns, *Abhand. Math.-Phys. cl. K. Sachs, Ges. Wiss.*, v. 21 (1895), p. 355. Cf. J. L. Synge, *Journal of the Optical Society of America*, v. 27 (1937), p. 75–82, 138–144.

independently rediscovered Hamilton's third characteristic function and named it the "eikonal" (or image function). The name stuck, and there grew up a school which recognized the fundamental value of Bruns' eikonal in practical geometrical optics, without understanding that it was precisely Hamilton's invention.

Hamilton carried his idea of characteristic or principal function over into dynamics. He showed that, in dynamics as well as in optics, this function satisfies two partial differential equations. The problem of solving the *ordinary* differential equations of dynamics was changed into that of solving two *partial differential equations*. (This opened up the theory of infinitesimal contact transformations.) The work excited the interest of Jacobi (1804–1855), who made a most significant contribution in showing how to use Hamilton's theory. But (such is the cynicism of fate!) Jacobi belonged less to the heroic age than Hamilton, and failed to see the psychology behind the theory. He regretted that Hamilon had darkened the issue by introducing *two* partial differential equations instead of *one*. Now the truth of the matter is this. If you are interested only in solving dynamical problems, one equation is better than two (the second equation is merely an encumbrance). But if your motive is to reduce the consideration of a dynamical system to that of *one single principal function*, then the second equation is fundamental. Jacobi's solutions of the single partial differential equation are useful artisans; they do not belon to the royal house of principal functions.

After Jacobi's contribution (which is certainly to be regarded as of prime importance) the partial differential equation—one of Hamilton's two—was often referred to as the Hamilton-Jacobi equation. Recently, there has been an unfortunate tendency[3] to drop the name of Hamilton, and call it "Jacobi's equation," thus completing the eclipse.

Among the highest authorities on dynamics, Hamilton's principal function is not as well known as might be wished. Thus G. D. Birkhoff in his Colloquium Lectures[4] does not use the idea, and refers to quite a different function as the Hamiltonian "principal function."

Despite the fact that Hamilton's early work deals ostensibly with physics (optics and dynamics), he has no claim to be regarded as a great physicist. He was a mathematician through and through. Nevertheless there is a greater appreciation of his work among mathematical physicists today than among pure mathematicians.[5] The

[3] Cf. Th. DeDonder, *Theorie invariantive du calcul des variations* (Paris, 1935), p. 207.

[4] G. D. Birkhoff, *Dynamical Systems* (New York, 1927), p. 52.

[5] Hamilton's name does not occur in the bibliography of M. Morse, *Calculus of Variations in the Large* (New York 1934).

Hamilton-Jacobi equation suddenly sprang into prominence, after nearly a century, as one of the cornerstones of quantum mechanics. Perhaps mathematicians have not yet realized the magnitude of Hamilton's contribution to the calculus of variations. Archaic it may be, but there are some ideas that never grow old, because they are essentially simple in concept. If Hamilton's ideas have given a strong stimulus to mathematical physics, may there not be hidden in his work (interpreted as pure mathematics) some pregnant ideas unfamiliar to modern students of the calculus of variations? This extraordinary man, who lived in the nineteenth century, seems to have written partly for the eighteenth and partly for the twentieth.

I have emphasized the fact that Hamilton's work may be regarded as belonging strictly to the calculus of variations. But it would be unwise to neglect the fact that optics and dynamics were its parents. In much of pure mathematics, the problems are set by the mathematician himself; in applied mathematics they are set by nature. This not only imposes a stimulating discipline, it gives also a scale of relative values among various problems, and dictates the special conditions which it is most interesting to impose.

Hamilton's work on the characteristic or principal function passed through phases of increasing generality. First, we have optics, in which the function is

$$V(x',y',z', x, y, z) = \int_{A'}^{A} v\left(x, y, z, \frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du}\right)du.$$

The integrand v is homogeneous of degree unity in the derivatives. It is liable to undergo abrupt changes across certain surfaces, and may be a multiple-valued function of its directional arguments. The case where v is constant with respect to changes in x, y, z (except for jumps across surfaces of discontinuity) is of particular importance. The extremals are then broken straight lines, not curves; differential equations are replaced by difference equations. The theory takes a form very different from the usual calculus of variations of the present day, and is naturally much richer in results, since there are no differential equations to solve. In this field, other characteristic functions, obtained from V by Legendre transformations, play an important part —in fact, they form the basis of the practical applications of Hamilton's method.

In dynamics, we have an integrand $L(x, y, t)$, where x stands for coordinates $x_i(i = 1, 2, \ldots N)$ and y for $y_i = dx_i/dt$. The principal function is

$$S(x',t', x, t) = \int_{A'}^{A} L(x, y, t)dt,$$

where A', A are the "points" (x', t'), (x, t), respectively. The function L usually has all desirable properties of smoothness, except for certain regions of singularity, where L goes to infinity.

The question naturally arises: Is it desirable to give one general form to the theory of the characteristic or principal function, in order to cover both optics and dynamics? It is hard to answer such a question once for all. Generality is advantageous in some ways, but not in others. It would be inadvisable to regard three-dimensional Euclidean geometry merely as a special case of N-dimensional Riemannian geometry, because generalizations are truly valuable only when they introduce sweeping simplicities.

Hamilton intended at one time to write a book on the "Calculus of Principal Relations," which would have carried out the generalization. This project, however, did not get beyond the preliminary stage. His notebooks[6] contain the general formula for the principal function

$$S = \underline{\int} \Phi(x_1, \ldots x_i, dx_1, \ldots dx_i),$$

where Φ is homogeneous of degree unity in the differentials, and the stroke under the integral sign indicates evaluation along an extremal. (Notice Hamilton's precision in notation, of which this is a small example.) He gave an interesting method of determining S by successive approximations.

To the reader who relies on this article for a sketch of the prequaternionic Hamilton, an apology is due. I believe that the motivating or guiding ideas of great scientists are of fundamental importance, and therefore I have tried to lay all possible stress on these ideas in the case of Hamilton. In doing this, I have slighted contributions which have brought Hamilton's name into greater prominence. Thus there is little doubt that the mathematical discovery of conical refraction in crystals was regarded in Hamilton's lifetime as his most brilliant achievement in optics. In dynamics, Hamilton's Principle

$$\delta \int L dt = 0,$$

the Hamiltonian H (total energy), and the canonical equations

$$\dot{q} = \partial H / \partial p, \quad \dot{p} = - \partial H / \partial q$$

are familiar to every student of dynamics. We get nearer to the heart of Hamilton's work if we write down the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0. \qquad (A)$$

[6] *The Mathematical Papers of Sir W. R. Hamilton*, Vol. II (Cambridge, 1940), p. 333.

I venture to think that very few indeed realize that Hamilton placed side by side with (A), as co-equal, the second partial differential equation (deplored by Jacobi)

$$-\frac{S}{\partial t'} + H\left(-\frac{\partial S}{\partial q'}, q'\right) = 0 \qquad (A')$$

We know very little of the psychology of mathematicians, who, of all scientists, present to their colleagues in public the most austere and unrevealing mask. Behind that mask there are quick jumps of intuition and much mental stumbling and confusion. As a rule, the smooth presentation of a finished paper shows neither the one nor the other. It is only in the intimacy of conversation that the real secrets are revealed. The extraordinary thing in the case of Hamilton is that he proclaimed (one might even say, shouted) at the beginning of each paper the motivating essence of his work—the characteristic or principal function, but those who read his work found in it everything but the central idea. Which shows the rashness of any mathematician who tries to impose an unfamiliar mental pattern on his age. Let us wait another century to see whether Hamilton or Jacobi had the subtler mind.

The Life of Hamilton

William Rowan Hamilton was born in Dublin at midnight August 3–4, 1805, and died September 2, 1865. He was of Irish stock, with (perhaps) a Scottish maternal grandmother. His father was Archibald Hamilton, a Dublin solicitor, and his mother's maiden name was Sarah Hutton. W. R. H. was fourth in a family of nine, four of whom died in childhood. His mother died in 1817 and his father (after marrying again) in 1819. The fact that he was then an orphan, at fourteen years of age, did not have such a great effect on him as might be expected, because at the early age of three his parents had handed him over to his uncle, James Hamilton, a clergyman living in the country. No explanation is given of this (to us) rather extraordinary procedure. It worked well. Uncle, James was a classical scholar (Archibald Hamilton had no university education) and proved worthy of his charge, in which he was helped by his sister (Aunt Sydney).

A very complete account of Hamilton's life was written by his contemporary, R. P. Graves (*Life of Sir William Rowan Hamilton*, three volumes, Longmans Green, London 1882–1889). A most valuable feature of this enormous work consists of letters reproduced in full—by Hamilton, to Hamilton, and about Hamilton. From these letters, and the comments of the biographer, we can reconstruct a de-

tailed picture of Hamilton's childhood. He was certainly a precocious child, but seems to have avoided certain objectionable features of precocity. Anyone interested in the details is advised to consult Graves' *Life*. I shall content myself by quoting Graves' summary (Vol. I, pp. 46–47) on Hamilton up to the age of nine:

"It will then be noted that, continuing a vigorous child in spirits and playfulness, he was at three years of age a superior reader of English, and considerably advanced in arithmetic; at four a good geographer; at five able to read and translate Latin, Greek, and Hebrew, and loving to recite Dryden, Collins, Milton, and Homer; at eight he has added Italian and French, and gives vent to his feelings in extemporized Latin, and before he is ten he is a student of Arabic and Sanscrit. And all this knowledge seems to have been acquired, not indeed without diligence, but with perfect ease, and applied, as occasion arose, with practical judgment and tact. And we catch sight of him, when only nine, swimming with his uncle in the waters of the Boyne. In this accomplishment he afterwards became a proficient."

The period 1816–1823 was spent in school in the country. His work there was directed towards a university career in the widest sense, and already at the age of seventeen he began original research in optics. In 1823 he entered Trinity College, Dublin, and had a distinguished undergraduate career. He found time, however, to write a paper under the title "On Caustics," and submitted it to the Royal Irish Academy for publication. In modern language, it dealt with congruences of straight lines and their singularities. The referees of the Academy rejected it in vague terms, and this manuscript had to wait over a century for publication.

This rebuff was, however, amply compensated by a signal distinction which came to Hamilton not long afterwards. Dublin University (which is coextensive with Trinity College, Dublin) controls Dunsink Observatory in the neighborhood of Dublin. In 1827 a vacancy occurred at the Observatory, and Hamilton became Andrews Professor of Astronomy and Royal Astronomer of Ireland, before receiving his B.A. degree—an appointment showing remarkable sagacity on the part of the Board of Trinity College.

It should be explained that Hamilton's reputation was such that his academic future was assured. But he had a choice between this professorship and a Fellowship in Trinity College. A Fellowship implied life membership of the collegiate corporation, and a considerable expenditure of time on elementary teaching or administration. There was also a monastic edict of celibacy. The astronomical position gave far greater freedom (since the routine operations of the Observa-

tory would be largely left to an assistant), but it did mean a measure of isolation from the academic life of the University. Perhaps the Board would have been wiser if they had created a special position for Hamilton in view of his quite extraordinary gifts—a position combining freedom for research with a closer contact with the University. Then, perhaps, Hamilton might have become the center of a school, rather than a magnificent isolated figure.

Indeed the proposal to translate Hamilton into the chair of Mathematics was made in 1843, when Hamilton was already at the height of his fame. Hamilton would himself have liked the change, and there is little doubt that its effect on Dublin University would have been very stimulating. Unfortunately, the proposal was not favorably received by the University authorities (Graves, *Life* Vol. II, p. 423), who would have required Hamilton to sit for an examination for Fellowship and to take holy orders!

From his appointment in 1827 to his death in 1865 there is a span of thirty-eight years. This span is divided by the year 1843, in which he invented quaternions. The sixteen years (1827–1843)—the pre-quaternion years—were devoted in the main to optics and dynamics. The twenty-two years (1843–1865) were given up almost wholly to quaternions.

There is not much to relate of his purely personal life. In 1833 he married Helen Maria Bayly. They had two sons and one daughter. As a negative fact of heredity, it may be remarked that great mathematical talent appeared neither in Hamilton's known forebears nor in his descendants.

The even tenor of his life was broken by occasional visits to England and Scotland—chiefly to meetings of the British Association for the Advancement of Science. He never visited the continent of Europe, nor did he come to America. But he received many distinctions abroad as well as at home.

In 1835 he was knighted at the meeting of the British Association in Dublin. From 1837 to 1846 he was President of the Royal Irish Academy. He was elected a Fellow of the American Academy of Arts and Sciences in 1832, corresponding member of the Russian Imperial Academy of Sciences in 1837, of the Royal Academy at Berlin in 1839, of the French Academy of Sciences in 1844. In 1863 the National Academy of Sciences was formed—the first truly national academy in the United States. Hamilton's name was placed at the head of the list of ten Foreign Associates elected by the Academy shortly after its foundation. The first official notification failed to reach Hamilton,

but a second letter telling him of his signal honor reached him a few months before his death.

My chief justification for attempting an appraisal of Hamilton rests on my experience during the years 1925–1930, when I was Professor of Natural Philosophy at the University of Dublin, and undertook the editing of the first volume of Hamilton's collected mathematical papers in collaboration with Professor A. W. Conway of University College, Dublin. This task was undertaken at the request of the Royal Irish Academy, which sponsored the publication of the collection as Cunningham Memoirs. Two volumes have now appeared, printed at the Cambridge University Press—Volume I (*Geometrical Optics*) in 1931 and Volume II (*Dynamics*) in 1940, the second volume being edited by Professors A. W. Conway and A. J. McConnell.

I would like to place on record the contribution that my eldest brother, E. H. Synge, made to this project behind the scenes. Any part I may have played in initiating the collected edition of Hamilton's papers is due primarily to him. It was he who first gave me an interest in Hamilton's early work, and his constant urgings goaded me on in the face of preliminary discouragements.

It is a rare privilege to study the original manuscripts of a great mathematician. Everyone who attempts to pursue mathematical research finds his first enthusiasms dulled by the tedium of attention to details. So many petty decisions have to be made—to choose this notation or that, this order of development or that order, to write coherently or to resort to odd jottings, to keep old notes or to throw them out. It is a cruel comfort to find that the great mathematician cannot transcend these human limitations.

Hamilton had great confidence in himself. He never seems to have debated as to whether what he was doing was important. He wrote down everything and kept the notes. Most of his notes were coherently and legibly written, and those selected for publication required little editing. He wrote a small clear hand, generally using large notebooks with dated entries. He seldom made an error. He liked long arithmetical calculations which he carried out with precision. Hamilton's two hundred-odd notebooks in the Library of Trinity College, Dublin, refute the idea that great originality and penetration in a mathematician imply a contempt for detail and arithmetical calculation. He liked to refer to himself in the words Ptolemy used of Hipparchus: ἀνὴρ φιλόπονος καὶ φιλαλήθης—a lover of labor and a lover of truth.

But it was too great a task for one man. Even his enormous energy could not keep pace with his ideas. If a mathematician makes some

addition to a theory already in existence and well understood, publication of the result suffices. But when a new idea of penetrating generality is introduced, it is not enough to trust its future to a few papers. All sorts of particular and elementary applications have to be worked out before the new idea is accepted into the circle of mathematical familiarity. Just at the time when Hamilton was getting ripe for such detailed work in the theory of the principal function, he became absorbed by quaternions and never returned to his early work. If only he had had the collaboration of some advanced students to handle details, his influence on succeeding generations of mathematicians might have been greater. Perhaps, on the other hand, such a collaboration was inconsistent with his temperament, and he preferred to be, as Wordsworth wrote of Newton,

"... a mind forever
Voyaging through strange seas of thought, alone."

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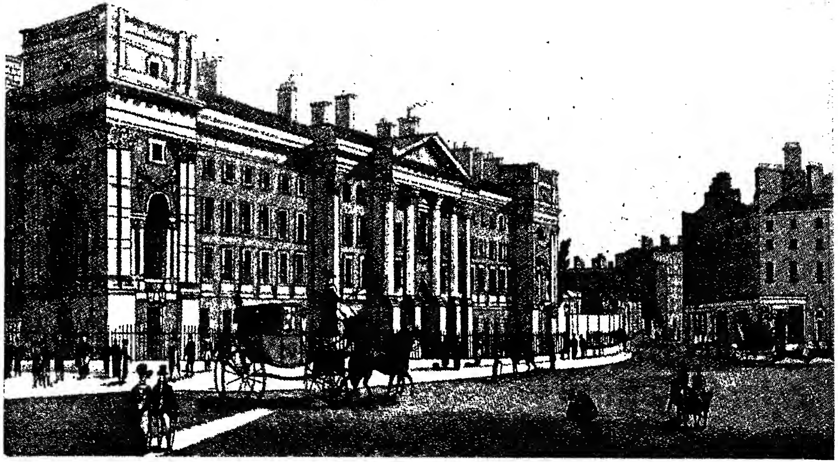

TRINITY COLLEGE A CENTURY AGO.

ALGEBRA'S DEBT TO HAMILTON

By C. C. MacDuffee

1. Introduction

Now that one hundred years have elapsed since Hamilton first brought out his celebrated paper on quaternions, it might be well to attempt an evaluation of his contributions to algebra. During Hamilton's lifetime and for a few decades after his death, there was much difference of opinion and some acrimony concerning the value of quaternion analysis. It did not prove to be the magic formula in geometry and mechanics that its inventor had dreamed it to be. For a while Hamilton's fame was in eclipse. But as the centenary of his discovery approached, a new appreciation of Hamilton developed, not among the geometers, but among the workers in algebra. For Hamilton's careful, detailed and logical criticisms of the foundations of algebra have been important steps in the development of modern abstract algebra.

Quaternions were not Hamilton's only contribution to algebra; they were not even his major contribution, although they were most spectacular. His presentation and logical justification of the complex number field as a theory of couples of real numbers introduced a fundamental concept into algebra, a concept which overshadows the concept of quaternion. Indeed, quaternions were a logical and relatively direct generalization of it, and once the theory of couples, or n-tuples, had been expounded, the discovery of quaternions and other linear algebras was bound to follow.

The theory of couples came to Hamilton at a relatively early age. His paper was presented to the Royal Irish Academy in 1833 when the author was but twenty-eight. I wish to quote from his introduction:[1]

"The study of algebra may be pursued in three very different schools, the practical, the philological, or the theoretical, according as algebra itself is accounted an instrument, or a language, or a contemplation;

[1] "Conjugate Functions and on Algebra as the Science of Pure Time," *Transactions of the Royal Irish Academy*, vol. xvii, pt. 2, p. 293–422. Dublin, 1835.

according as ease of operation, or symmetry of expression, or clearness of thought is eminantly prized and sought for. The practical person seeks a rule which he may apply, the philological person seeks a formula which he may write, the theoretical person seeks a theorem on which he may meditate. . . .

"These remarks have been premised that the reader may more easily and distinctly perceive what the design of the following communication is. . . . The thing aimed at is to improve the *science*, not the art nor the language of algebra. The imperfections sought to be removed are confusions of thought and obscurities or errors of reasoning. . . . And that confusions of thought and errors of reasoning still darken the beginnings of algebra is the earnest and just complaint of sober and thoughtful men who in a spirit of love and honor have studied algebraic science, admiring, extending, and applying what has been already brought to light, and feeling all the beauty and consistence of many a remote deduction, from principles which yet remain obscure and doubtful. . . ."

It must be remembered that these words were written more than a century ago when algebra was still in the Dark Ages. Algebra still contains confusions of thought, but in 1833 there was little but confusion. It is apparent that Hamilton had a remarkably clear notion of where his difficulties lay, for his preface continued:

"It requires no peculiar scepticism to doubt, or even to disbelieve, the doctrine of negatives and imaginaries, when set forth (as it has commonly been) with principles like these: that a *greater magnitude may be subtracted from a less*, and that the remainder is *less than nothing;* that *two negative numbers*, or numbers denoting magnitudes each less than nothing, may be *multiplied* the one by the other, and that the product will be a *positive* number, or a number denoting a magnitude greater than nothing; and that the *square* of a number, or the product obtained by multiplying that number by itself, is therefore *always positive*, whether the number be positive or negative, yet that numbers, called imaginary, can be found or conceived or determined, and operated on by all the rules of positive and negative numbers, as if they were subject to those rules, *although they have negative squares*, and must therefore be supposed to be themselves neither positive nor negative, nor yet null numbers, so that the magnitudes which they are supposed to denote can neither be greater than nothing, nor less than nothing, nor even equal to nothing. It must be hard to found a science on such grounds as these. . . ."

It is startling to notice that these lines of Hamilton, written one hundred and eleven years ago, summarize so perfectly the confusion

which quite generally still persists in the teaching of elementary algebra in our secondary schools.

2. The Paper of 1833

At the age of twenty-eight, Hamilton was under the influence of Newton, and attempted to found his theory of algebra on the time continuum as Newton had founded his fluxions. At this stage in his development Hamilton did not think of definitions and postulates as we do today, as the basic hypotheses of our mathematical system, but, on the contrary, he felt called upon to appeal to some physical concept for their justification. The continuum of time was the concept which he chose. As Hamilton thought more deeply about these matters, he was led to the abstract viewpoint,[2] but to the end of his career he felt it necessary to appeal to the physical universe in order to justify his abstract notions in the eyes of his contemporary scientists.

Hamilton supposed the positive integers and their elementary properties to be known. He considered an "equidistant series of moments"

$$\ldots E''\, E'\, E\, A\, B\, B'\, B'' \ldots$$

each letter representing an instant or moment of time such that the intervals of time between two successive moments were all equal to one another. Some moment such as A was selected "as a standard with which all the others are to be compared" and was called the *zero-moment.* One of the moments near to A (such as B) was called the *positive first.* The operator or step by which one passed from any moment to the next moment to the right was denoted by a, so that

$$B = a + A,\, B' = a + B = a + a + A = 2a + A,\, \ldots .$$

The operator by means of which one passed from any moment to the next moment to the left was denoted by θa. Thus

$$A = \theta a + B,\, E = \theta a + A,\, \ldots .$$

The series of steps from the zero-moment to another moment were now denoted by

$$\ldots 3\theta a,\, 2\theta a,\, 1\theta a,\, 0a,\, 1a,\, 2a,\, 3a,\, \ldots .$$

This series of multiple steps was formed by combining the symbol of the base a with the following series of ordinal symbols,

$$\ldots 3\theta,\, 2\theta,\, 1\theta,\, 0,\, 1,\, 2,\, 3,\, \ldots .$$

[2] See the letter to John T. Graves to which reference is made later.

If ξ, ν, μ, ω are ordinals of this series, Hamilton proved to his own satisfaction such properties as

$$\mu + \nu = \nu + \mu, \quad \mu\nu = \nu\mu, \quad \theta\theta = 1,$$
$$\xi(\mu\nu) = (\xi\mu)\nu, \quad (\nu' + \nu)\mu = \nu'\mu + \nu\mu.$$

Clearly θ, or 1θ, is the ordinal -1.

Rational fractions were introduced similarly. The step $\nu/\mu a$ was that step which, when applied μ times, was equivalent to the step νa. All the properties of the rationals followed readily. Hamilton stated that "The fractional sign 1/0 denotes an impossible act."

Hamilton's introduction of the irrationals was surprisingly modern. He first considered the existence of a square root of a positive number or, as he stated it, the introduction of a mean proportional between two positive numbers. He stated that if $a > n'/m'$ whenever $n'^2/m'^2 < b$ and if $a < n''/m''$ whenever $n''^2/m''^2 > b$, then $a = \sqrt{b}$. The existence of such an a was argued in a series of four lemmas.

Powers, roots, and logarithms were then discussed. This ended what Hamilton called the "preliminary and elementary essay," the first part of his 129-page paper.

3. Introduction of the Complex Numbers

The second part of the paper, entitled "Theory of Conjugate Functions or Algebraic Couples," looks as if it might have been written a little later than the preliminary part. This is quite possible, for two years elapsed between the reading of the paper and its publication. At any rate Hamilton's point of view in this second paper seems to be slightly more advanced. It is true that he still based his fundamental concepts on the time series, but the postulational approach was clearly developing.

I cannot refrain from quoting a paragraph from the early part (p. 403) of this paper:

"Proceeding to operations upon number-couples, considered in combination with each other, it is easy now to see the reasonableness of the following definitions, and even their necessity, if we would preserve in the simplest way, the analogy of the theory of couples to the theory of singles:

$$(b_1, b_2) + (a_1, a_2) = (b_1 + a_1, b_2 + a_2);$$
$$(b_1, b_2) - (a_1, a_2) = (b_1 - a_1, b_2 - a_2);$$
$$(b_1, b_2)(a_1, a_2) = (b_1, b_2) \times (a_1, a_2) = (b_1a_1 - b_2a_2, b_2a_1 + b_1a_2);$$

$$\frac{(b_1,\, b_2)}{(a_1,\, a_2)} = \left(\frac{b_1a_1 + b_2a_2}{a_1^2 + a_2^2},\, \frac{b_2a_1 - b_1a_2}{a_1^2 + a_2^2}\right).$$

Were these definitions even altogether arbitrary, they would at least not contradict each other, nor the earlier principles of Algebra, and it would be possible to draw legitimate conclusions, by rigorous mathematical reasoning, from premises thus arbitrarily assumed: but the persons who have read with attention the foregoing remarks of this theory, and have compared them with the Preliminary Essay, will see that these definitions are really *not arbitrarily chosen*, and that though others might have been assumed, no others would be equally proper."

It is interesting to note the trepidation with which Hamilton presented this wholly logical development of the complex number field. He was not quite prepared to say in the modern manner, "These are my definitions and you have no right to question their propriety. That they are happily chosen will appear when their implications are investigated, for the theory of couples subject to the operations here defined will prove to be abstractly identical with the theory of those complex numbers whose logical foundation is otherwise so insecure."

Hamilton seems to have recorded his thoughts much as they came into his mind, and his thoughts were logical and extremely detailed. The paper under consideration contains a superabundance of detail judged by present-day standards, and I will not burden the reader with it. But toward the end there is a paragraph which is significant. The capitals and italics are characteristic of Hamilton's style and indicate the intensity of his personality.

"In the THEORY OF SIMPLE NUMBERS, the symbol $\sqrt{-1}$ is *absurd*, and denotes an IMPOSSIBLE EXTRACTION, or a merely IMAGINARY NUMBER; but in the THEORY OF COUPLES, the same symbol $\sqrt{-1}$ is *significant*, and denotes a POSSIBLE EXTRACTION, or a REAL COUPLE, namely the *principal square-root of the couple* $(-1, 0)$. In the latter theory, therefore, though not in the former, this sign $\sqrt{-1}$ may properly be employed; and we may write, if we choose, for any couple (a_1, a_2), whatever,

$$(a_1,\, a_2) = a_1 + a_2\sqrt{-1}."$$

4. Quaternions

In the eyes of Hamilton's contemporaries, his most spectacular accomplishment in algebra was unquestionably his discovery, just a

hundred years ago, of the algebra of real quaternions. This was such an unorthodox mathematical system, with its non-commutative multiplication, that it immediately attracted the attention of mathematicians the world over. Hamilton's reputation was by this time so secure that he did not have to undergo very much of the unreasoned criticism which is usually meted out to authors of unorthodox discoveries. Indeed, it is more than likely that quaternions were accepted in quarters where they were not thoroughly understood.

Hamilton's paper of 1833 on the theory of couples was the acknowledged ancestor of his theory of quaternions. It was very natural that the success of the theory of couples in establishing the foundations of the complex number field should suggest to the discoverer that new and unexplored regions lay in the theory of ordered sets. While one may contend with some justification that Cauchy's development of the complex numbers as polynomial residues of $x^2 + 1$ is in some respects more elegant than Hamilton's theory of couples, it cannot be denied that Hamilton's approach has wider possibilities in its generalizations. Both methods are now fundamental in abstract algebra. Cauchy's method is used to show the existence of algebraic extensions of fields, while Hamilton's method leads to the theory of linear algebras.

But there were many serious problems to be solved before the theory of quaternions could arise from the theory of quadruples. The broad outlines of the theory were there, but the details were lacking. It would be most interesting if Hamilton had left a record of his failures and futile investigations before he succeeded in finding what we now know to be the only division algebra in four units over the real field.

The problem which Hamilton set for himself was the extension to space of that theory of complex numbers which works so beautifully in the plane. This is a problem which seems to occur to every serious student of mathematics at some stage in his career, usually while he is yet an undergraduate. What would seem to be the obvious extension, namely, the construction of an algebra in three units to correspond to the three spatial dimensions, is now known to be impossible, for there is no real associative division algebra in three units. But this fact was not known in Hamilton's day, and inevitably he must have spent a great deal of time pursuing this goal before he became convinced of its unattainability.

We know from the statements of Hamilton's son that the birth of quaternions was a trying experience for Hamilton and for all his family. He would sit brooding in his office neglecting to eat unless food was brought to him, striving for that which his intuition told him

must be possible, but which he could not see. As he came to breakfast in the morning, his family would join in asking, "Have you succeeded in multiplying quaternions yet?"

The question might well be asked, what induced Hamilton to attempt to find an algebra in four units after it became evident that none existed in three units which would suit his purposes. I think a clue to this is to be found in his paper. Euler had discovered that the product of two sums of four squares can be written as a sum of four squares. Now the corresponding theorem concerning the product of sums of two squares is equivalent to the fact that the norm of the product of two complex numbers is equal to the product of their norms. Without much doubt, Hamilton caught the hint that there might exist an algebra in four units possessing the same norm property.

Even after he had come thus far, Hamilton had serious problems before him. A non-commutative multiplication was in that day unthinkable, mainly because the "ordinary operaitons" were taken for granted and such fundamental ideas as the commutative law had not even been formulated. Hamilton's ability to proceed in the face of the failure of this law must be considered a mark of genius.

Still basing his foundations on the time series, Hamilton called a *momental quaternion* a set

$$(A_0, A_1, A_2, A_3)$$

of four moments of time. Two such quaternions were called equal only if corresponding moments were equal. By an argument similar to the one used in the paper of 1833 for couples, he arrived at the quaternion $q = (a, b, c, d)$ of numbers.

Just as in the theory of couples the operator i had the property of changing the couple (a_1, a_2) into the couple $(-a_2, a_1)$ so in quaternion theory the three operators i, j, and k are such that

$$iq = (-b, a, -d, c),$$

$$jq = (-c, d, a, -b),$$

$$kq = (-d, -c, b, a).$$

By defining $(ij)q$ to mean $i(jq)$, Hamilton arrived at his famous multiplication table

$$i^2 = j^2 = k^2 = -1, ijk = -1.$$

Hamilton now made a fresh start and asked the reader to consider sets of n-tuples

$$q = (a_0, a_1, \ldots, a_{n-1})$$

and n operations or *simple derivatives* $\times_r$ such that

$$\times_r q = (b_0, b_1, \ldots, b_{n-1})$$

where b_i is a linear combination $b_i = \sum c_{ij} a_j$ with fixed coefficients of $a_0, a_1, \ldots, a_{n-1}$. The linear operator

$$\omega = m_1 \times_1 + m_2 \times_2 + \ldots + m_{n-1} \times_{n-1}$$

where the m's are numbers was called a *complex derivative*. If

$$\omega q = q',$$

then Hamilton wrote $q'/q = \omega$.

The way in which Hamilton approached the associative law was complicated. He set up n^4 equations which were necessary and sufficient in order that for any two complex operators ω_1 and ω_2, and any n-tuple q, it would be true that

$$\omega_2(\omega_1 q) = (\omega_2 \omega_1) q$$

where $\omega_2 \omega_1$ was independent of q.

Obviously if ω_3 is another operator,

$$(\omega_3 . \omega_2\omega_1)q = \omega_3(\omega_2\omega_1 q) = \omega_3[\omega_2(\omega_1 q)] = [\omega_3\omega_2](\omega_1 q) = (\omega_3\omega_2 . \omega_1)q,$$

so that this condition is equivalent to the associative law. Hamilton was unable to solve these n^4 conditions in the case $n = 4$, but he showed that the operarors i, j, and k, and of course the operator 1, satisfy them. Thus, without determining all linear associative algebras in four units, he showed that his quaternions were such an algebra.

It is clear that Hamilton considered his new mathematical entity, the quaternion, as the ratio of two quadruples of numbers. If in the quadruples (a_0, a_1, a_2, a_3) and (b_0, b_1, b_2, b_3) it is true that $a_i = kb_i$, then the ratio $(a_0, a_1, a_2, a_3)/(b_0, b_1, b_2, b_3)$ is the ordinary number k. If this proportionality relation does not hold, Hamilton conceived the ratio to be the new number, the quaternion.

In this paper of 1843, Hamilton found the formula

$$(w + ix + jy + kz)^{-1} = (w^2 + x^2 + y^2 + z^2)^{-1} (w - ix - jy - kz)$$

and showed that the reciprocal of a product is equal to the product of the reciprocals in reverse order. He recognized the importance of the norm

$$\mu^2 = w^2 + x^2 + y^2 + z^2$$

and proved that the norm of the product was equal to the product of

the norms. In this paper he proved that every quaternion is a root of a quadratic equation with real coefficients. This result is the origin of the famous Hamilton-Cayley theorem in the theory of matrices.

The paper closed with a remarkable analogue of the De Moivre theorem for complex numbers and the beginnings of the applications of quaternions to the geometry of space. At the end of the article there was printed a letter from Hamilton to John T. Graves, Esq., a fellow academician. This letter was written for a mathematician in whose ability and sympathy Hamilton had confidence, and its exposition was superior to that of the article which it followed. I quote:

"The Germans often put i for $\sqrt{-1}$, and therefore denote an ordinary imaginary quantity by $x + iy$. I assume three imaginary characteristics or units, i, j, k, such that each shall have its square $= -1$, without any one being the equal or the negative of any other. . . . And I assume (for reasons explained in my first letter) the relations

$$ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

Such being my fundamental assumptions which include (as you perceive) the somewhat strange one that the order of multiplication is not, in general, indifferent."

In the quaternion $q = w + ix + jy + kz$, let

$$\mu = \sqrt{w^2 + x^2 + y^2 + z^2}$$

and determine the angles θ, φ and ψ by the relations

$$\sin \psi = \frac{z}{\sqrt{y^2 + z^2}}, \qquad \cos \psi = \frac{y}{\sqrt{y^2 + z^2}},$$

$$\cos \varphi = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \qquad \cos \theta = \frac{w}{\sqrt{w^2 + x^2 + y^2 + z^2}}.$$

That is,

$$x = \mu \sin \theta \cdot \cos \varphi,\ y = \mu \sin \theta \cdot \sin \varphi \cdot \cos \psi,\ z = \mu \sin \theta \cdot \sin \varphi \cdot \sin \psi,$$
$$w = \mu \cos \theta,$$

so that one may write

$$q = \mu \cos \theta + \mu \sin \theta (i \cos \varphi + j \sin \varphi \cdot \cos \psi + k \sin \varphi \cdot \sin \psi).$$

The angle θ is called the amplitude.

There are special cases of this "polar form" of the quaternion which need special attention. If $y = z = 0$, $x \neq 0$, $\cos \psi$ and $\sin \psi$ are indeterminate; but $\cos \varphi = 1$ and $\sin \varphi = 0$ so that we can in this case write

$$q = \mu(\cos\theta + i\sin\theta),$$

which is the usual polar form of the complex number $w + ix$. But if $x = y = z = 0$, $w \neq 0$, $\cos\theta = 1$, $\sin\theta = 0$, and $q = \mu = w$. If $x = y = z = w = 0$, $\mu = 0$ so that $w = 0$.

The analogue of the De Moivre theorem is embraced in the result that q^n where n is an integer is obtainable from q by replacing μ by μ^n and θ by $n\theta$. For the exponent $1/n$ we have n determinations of θ,

$$\frac{\theta + 2p\pi}{n},\ p = 0, 1, \ldots, n - 1$$

so that, unless φ or ψ is indeterminate, a quaternion has just n distinct nth roots. If $y = z = x = 0$, $w \neq 0$ so that φ and ψ are indeterminate and $\theta = 0$, it will usually be true that $(\theta + 2p\pi)/n$ will not be zero so that for all values of φ and ψ we get values of $q^{1/n}$, which are therefore infinite in number. The only exception is for $n = 2$.

5. Later Work on Quaternions

It is not the purpose of this note to go into detail concerning Hamilton's later work on quaternions, to which he devoted most of the last twenty years of his life. His *Lectures on Quaternions* was published in 1853, and the large posthumous two-volume work, *Element of Quaternions*, was brought out in 1866 by Hamilton's son, William Edwin Hamilton.

Hamilton's enthusiasm for and belief in his quaternions was unbounded. He fully believed that his discovery was quite equal in practical importance to Newton's discovery of the calculus. He developed applications to geometry and mechanics and introduced into mathematics an elaborate variety of terms such as vector, scalar, tensor, versor, etc., to implement his theory. It was Hamilton's belief that his quaternions would prove to be the key that would unlock all the mysteries of geometry and mathematical physics. He had many followers such as Tait and Joly who loyally carried his banner into the front lines of all current scientific developments.

But Hamilton and his followers were thoroughly defeated in their attempts to supplant analysis as developed by continental mathematicians by quaternion analysis. Quaternions did not win a battle. There seemed to be no problem where it could be definitely said that quaternions furnished an easier or more natural approach than other methods. In fact, the tendency of British mathematicians to favor quaternions probably was a factor in their temporary loss of leadership in applied mathematics.

In the hands of Willard Gibbs quaternion analysis was simplified into vector analysis, and in this direction lay its greatest contribution to mathematical physics.

After the death of Hamilton and most of his loyal followers, the full worth of his mathematical labors became apparent. That Hamilton made a truly great mathematical discovery is now denied by none, but its importance did not lie in any applications which might be found for it in mathematical physics. Its real value was in its effect upon algebra.

One is inclined at this point to think of the parallel cases of Sophus Lie who labored a lifetime to apply his theory of continuous groups to the solution of differential equations, and of Kummer who discovered ideals in his attempts to prove Fermat's Last Theorem. Each man had a gold mine but was disappointed because he could not induce it to produce oil.

In this article I have endeavored to show that Hamilton was one of the most important of the early founders of modern abstract algebra. His paper of 1833 was devoted, as he said in his preface, "to the science, not the art nor the language of algebra," and his paper of 1843 was written in a similar vein. The point of view expressed in these papers is surprisingly modern, his logical justification of the complex numbers is elegant, and the use of n-tuples of numbers which he introduced is in wide use today in many branches of algebra.

One can scarcely resist attempting to draw a moral from Hamilton's experience with quaternions. His early paper was pure abstract mathematics. It was written just as the author thought, and his thoughts at that time were uncontaminated by a desire to be "useful." But it is quite likely that Hamilton's contemporaries kept inquiring, "What is the use of all this? What is it good for? Can it be applied to physics?" For then, as now, a branch of mathematics was considered by some people to be of value only to the extent that it could be applied in physics even though the physics to which it might be applied was fully as abstract and far from engineering practice as was the mathematics. Possibly it was this atmosphere, facing Hamilton at Dublin, which persuaded him to make a "practical tool" of his quaternions. At any rate the attempt was unsuccessful, and as time passes it becomes more and more evident that Hamilton's reputation in algebra is going to be founded upon his truly profound researches into the foundations of algebra. And I believe that if Hamilton were alive today, he would be content to have it rest there.

THE UNIVERSITY OF WISCONSIN

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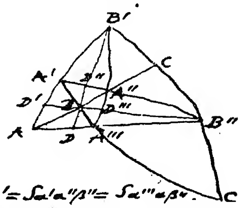
Let $\alpha\,\alpha'\,\alpha''\,\alpha'''$ be, as in Lect. art. 522, any four diverging vectors; and let $a = S\alpha'\alpha''\alpha'''$, $a' = S\alpha''\alpha'''\alpha$, $a'' = S\alpha'''\alpha\alpha'$, $a''' = S\alpha\alpha'\alpha''$.
Then, as in Lect. art. 523, $V(V\alpha'\alpha''.\,V\alpha'\alpha) = a''\alpha'' - a'''\alpha''' = a'\alpha' - a\alpha$;
and since $V(V\alpha\alpha'''.\,V\alpha''\alpha') = a'''\alpha''' - a\alpha = a''\alpha'' - a'\alpha'$; so that

$a\alpha + a''\alpha'' = a'\alpha' + a'''\alpha'''$. Let



$$\begin{cases} \beta = a\alpha + a''\alpha'' = a'\alpha' + a'''\alpha'''; \\ \beta' = a''\alpha'' - a'''\alpha''' = a'\alpha' - a\alpha; \\ \beta'' = a'''\alpha''' - a\alpha = a''\alpha'' - a'\alpha'. \end{cases}$$

We shall then have the coplanarities,

$0 = S\alpha\alpha''\beta = S\alpha'\alpha'''\beta = S\alpha\alpha'\beta' = S\alpha''\alpha'''\beta' = S\alpha'\alpha''\beta'' = S\alpha'''\alpha\beta''$.

Also let $\gamma = \beta'' + \beta' = a''\alpha'' - a\alpha$, $\gamma' = \beta'' - \beta' = a'''\alpha''' - a'\alpha'$;
then $0 = S\alpha\alpha''\gamma = S\beta'\beta''\gamma = S\alpha'\alpha'''\gamma' = S\beta'\beta''\gamma'$;

γ is intersection of $\beta'\beta''$ with plane of $\alpha\alpha''$;
γ' . . — — — — — — $\alpha'\alpha'''$;
β . . . . . . $\alpha\alpha''$ . . . . — $\alpha'\alpha'''$;

and we have the 3 harmonic pencils; $(\gamma\gamma';\ \beta', \beta'')$;
$(\beta, \gamma;\ \alpha, \alpha'')$; $(\beta, \gamma';\ \alpha', \alpha''')$; so that the 3 diagonals of a spherical (or plane) quadrilateral cut each other harmonically.

Let $\delta = a'''\alpha''' + a\alpha$, $\delta' = a\alpha + a'\alpha'$, $\delta'' = a'\alpha' + a''\alpha''$, $\delta''' = a''\alpha'' + a'''\alpha'''$;
then $\delta = \beta - \beta'$, $\delta' = \beta - \beta''$, $\delta'' = \beta + \beta'$. $\delta''' = \beta + \beta''$;
and we have then 2 other harmonic pencils, $(\beta, \beta';\ \delta, \delta'')$, $(\beta, \beta'';\ \delta', \delta''')$.

Let $\varepsilon = x\alpha + x''\alpha''$; $\varepsilon' = x'\alpha' + x'''\alpha'''$; $\zeta = y'\beta' + y''\beta''$;
$S\varepsilon\varepsilon'\zeta = 0$; $\varepsilon'' = x\alpha - x''\alpha''$; $\varepsilon''' = x'\alpha' - x'''\alpha'''$; $\zeta' = y'\beta' + y'''\beta''$;
$S\varepsilon''\varepsilon'''\zeta' = 0$; shall we have $y''' = -y''$? Yes. See below.

$0 = xx'S\alpha\alpha'\zeta + xx'''S\alpha\alpha'''\zeta + x''x'S\alpha''\alpha'\zeta + x''x'''S\alpha''\alpha'''\zeta$;
but $S\alpha\alpha'\zeta = y'S\alpha\alpha'\beta' + y''S\alpha\alpha'\beta'' = y''S\alpha\alpha'\beta'' = y''a'''a''$;
$S\alpha\alpha'''\zeta = -y'a''a'$; $S\alpha''\alpha'\zeta = y'a'''a$; $S\alpha''\alpha'''\zeta = -y''aa'$;
$\therefore\ 0 = xx'y''a'''a'' - xx'''y'a''a' + x''x'y'a'''a - x''x'''y''aa'$;

$\therefore\ \dfrac{y''}{y'} = \dfrac{a'''ax'x'' - a'a''xx'''}{aa'x''x''' - a''a'''xx'}$. Changing x'' & x''' to $-x''$ and $-x'''$, $\dfrac{y''}{y'}$ is changed to $-\dfrac{y'''}{y'}$; $\therefore\ y''' = -y''$, as expected.

We may make $y' = aa'x''x''' - a''a'''xx'$, $y'' = a'''ax'x'' - a'a''x''x$
Then $\zeta = (a'\alpha' - a\alpha)(aa'x''x''' - a''a'''xx')$
$+ (a''\alpha'' - a\alpha)(a'''ax'x'' - a'a''x''x)$

~~$+\ldots$~~ $+ aa''e\,e + a'a'''e'e' + (a'\alpha' + a''\alpha'' - a\alpha - a'''\alpha''')\,xx'$;
$a'e'x''' + aex'' = a'''ax'x'' - a'a''xx'''$; $a'''ex' + aex''' = aa'x''x''' - a''a'''xx'$;
$e' = \frac{aa''}{a} a''x$; $e = a'''x' + a'x'''$; $\zeta = aa''ee + a'a'''e'e'$.

A Page from One of Hamilton's Notebooks
Courtesy of the Royal Irish Academy

AN ELEMENTARY PRESENTATION OF THE THEORY OF QUATERNIONS

By F. D. Murnaghan*

When Hamilton discovered and presented to the world his theory of quaternions he was so far ahead of his time that the theory was little understood. It was regarded as extraordinarily difficult and for three quarters of a century the standard example of something "out of this world" and not for the general was a lecture on quaternions; it was only when Einstein proposed his General Theory of Relativity that the reporter turned to this modern illustration and forgot about quaternions. Now Hamilton was one of the greatest mathematicians of all time and so knew that the really important things in mathematics are fundamentally simple; just as Einstein asserts that Nature is essentially simple and direct. But mathematics, like Nature, guards dearly the secrets of her working and we are often forced to watch "through a glass darkly." We propose in the present paper to take away this glass as far as quaternions are concerned and to present the theory in a manner which might meet the approval of its great discoverer.

1. De Moivre's Theorem. Hamilton was led to his discovery of quaternions by his appreciation of the significance of the ordinary imaginary unit i of complex number theory as an *operator* in the domain of plane vectors. Using the convenient notation of matrices any plane vector x may be presented as a 2×1 (i. e., a two-row, one-column) matrix:

$$x =$$

and, in particular, i is the 2×1 matrix Then ix is the plane vector obtained by rotating x through a positive quarter cycle so that

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$$ix = \begin{pmatrix} -x^2 \\ x^1 \end{pmatrix} = Ix$$

where I is the 2×2 alternating matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We say that the 2×2 matrix I is a *realization* of the imaginary unit i when we regard i as an operator (or *multiplier*); whilst the 2×1 matrix $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a realization of i when we regard i as a plane vector (or *adder*). The 2×2 realization has the advantage over the 2×1 realization that we can use it for both purposes. In other words, the proper representation of the *complex number* x (i. e., the element x of the algebra of complex numbers and not merely the plane vector x) is the 2×2 matrix

$$x \cdot \quad X = \begin{matrix} x^1 & -x^2 \\ x^2 & x^1 \end{matrix} = x^1 E_2 + x^2 I$$

where E_2 is the 2×2 unit matrix.

One of the most fundamental relations in the algebra of complex numbers is expressed by de Moivre's formula

$$\cos\theta + i \sin\theta = e^{\theta i}.$$

When i is realized by the 2×2 alternating matrix I this relation appears in the form

$$O = e^{\theta I}$$

where O is the 2×2 rotation matrix ($= 2 \times 2$ orthogonal matrix with determinant $+1$)

$$O \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The symbol $e^{\theta I}$ is merely an abbreviation for the infinite series $E_2 + \theta I + \frac{\theta^2 I^2}{2!} +$ In general, if A is any $n \times n$ matrix we use the symbol e^A to indicate the infinite series $E_n + A + \frac{A^2}{2!} + \ldots$ (it being easy to show that this series converges for every A). Hamilton knew all this although he did not know the matrix realization (for matrix theory was unknown when Hamilton discovered quaternions; we think it fair to say that it was the explosive effect of Hamilton's discovery that

forced the discovery of matrices). He then asked himself the question: is all this confined to *plane* vectors, or is there some generalization of complex numbers which may furnish for vectors in three-dimensional space an algebra similar to the algebra of complex numbers? As far as de Moivre's formula is concerned, there is an obvious generalization to three, and in fact to n dimensions. We first observe that if A and B are any $n \times n$ matrices the fact that A and B do not, in general, commute, i. e., that $AB \neq BA$, in general, makes invalid the binomial formula:

$$(A + B)^m = A^m + mA^{m-1}B + \ldots + B^m.$$

However, if A and B do happen to commute, this formula is valid for every positive integral m and this implies that $e^A e^B = e^{A+B}$. If A is an *alternating* $n \times n$ matrix and we denote by B its transpose A' we have $B = A' = -A$ so that B and A commute. Since $A + B$ is the zero matrix it follows that $e^A e^{A'} = E_n$ and since $e^{A'}$ is the transpose of e^A it follows that e^A is an *orthogonal* $n \times n$ matrix O. It is, in fact, a rotation matrix since the sum of the diagonal elements of A (i. e., Tr A) is zero and since $\det e^A = e^{\mathrm{Tr} A}$. Hence we have the result: *If A is any alternating $n \times n$ matrix e^A is an $n \times n$ rotation matrix O.* The converse of this result is equally true: *If O is any $n \times n$ rotation matrix we may write it in the form e^A where A is an alternating matrix.* We content ourselves with the remark that this important result is a mere corollary of de Moivre's Theorem (for 2×2 matrices) once we know that if O is any $n \times n$ rotation matrix there exists an $n \times n$ rotation matrix R such that ROR' is the direct sum of rotation matrices each of dimension $\leqslant 2$. We shall refer to the relation

$$O = e^A; \ O \text{ a rotation matrix, } A \text{ an alternating matrix}$$

as the *generalized de Moivre Theorem.*

Since every $n \times n$ matrix satisfies an algebraic equation of degree $\leqslant n$ we may transform the *infinite series* e^A into a polynomial of degree $<n$. In the case $n = 2$ this transformation is very simple; in fact $A = \theta I$, and since $I^2 + E_2 = 0$, $A^2 + \theta^2 E_2 = 0$ and so $e^A = \left(1 - \frac{\theta^2}{2!} + \ldots\right) E_2 + \left(\theta - \frac{\theta^3}{3!} + \ldots\right) I = \cos\theta + \sin\theta I$ which is de-Moivre's original formula. When $n = 3$ we may proceed as follows. Write A in the form

$$A = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$$

and use the three-dimensional vector (p, q, r) to identify A. If θ is the magnitude of this vector, so that $p^2 + q^2 + r^2 = \theta^2$, we have $A^3 + \theta^2 A = 0$ as the algebraic equation, of degree 3, which is satisfied by A. On availing ourselves of this equation we may eliminate from e^A all powers of A of degree >2 and so $O = e^A = c_0 E_3 + c_1 A + c_2 A^2$. Instead of actually attempting this elimination we may determine the coefficients c_0, c_1 and c_2 by the following artifice. There exists a rotation matrix R such that

$$RAR' = \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $e^{RAR'} = c_0 E_3 + c_1(RAR') + c_2(RAR')^2$ since RAR' satisfies precisely the same equation of degree 3 as does A. Since

$$e^{RAR'} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (RAR')^2 = \begin{pmatrix} & 0 & 0 \\ 0 & -\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have

$$c_0 = 1; \; c_0 - c_2\theta^2 = \cos\theta; \; c_1\theta = \sin\theta$$

so that

$$O = E_3 + \frac{\sin\theta}{} A + \frac{1 - \cos}{\theta^2}$$

(when $\theta = 0$, $A = 0$ and $O = E_3$). On writing

$$p = l\theta; \; q = m\theta; \; r = n\theta; \; A = J\theta,$$

so that (l, m, n) is the unit vector along the vector (p, q, r), we have

$$O = E_3 + \sin\theta J + (1 - \cos\theta)J^2; \; J^3 + J = 0.$$

This furnishes the familiar specification of the 9 elements of any 3×3 rotation matrix in terms of four parameters θ, l, m, n where $l^2 + m^2 + n^2 = 1$. If we write

$$l \sin\frac{\theta}{2} = \lambda; \; m \sin\frac{\theta}{2} = \mu; \; n \sin\frac{\theta}{2} \quad \nu; \; \cos\frac{}{2} = \rho$$

so that $\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1$, we obtain

$$O = E_3 + 2\rho K + 2K^2; \; K = \sin\frac{\theta}{2} J = \begin{pmatrix} 0 & -\nu & \mu \\ \nu & 0 & -\lambda \\ -\mu & \lambda & 0 \end{pmatrix}$$

This furnishes, in compact form, the parametric representation of the elements of any 3×3 rotation matrix in terms of four parameters λ, μ, ν, ρ connected by the relation $\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1$. This parametric representation was furnished by Euler some seventy years before the discovery of quaternions, and now that we look back the essential fact seems crying aloud for recognition. Whilst it is true that there are only three degrees of freedom in the domain of 3×3 rotation matrices, it is not nearly as convenient to use *three independent* parameters as to use *four* dependent parameters $(\lambda, \mu, \nu, \rho)$ connected by the single relation $\lambda^2 + \mu^2 + \nu^2 + \rho^2 = 1$. Furthermore the quadratic nature of this relation, and of the formulae which express the elements of O in terms of $(\lambda, \mu, \nu, \rho)$, is worthy of serious thought. When θ is increased by 2π the parameters $(\lambda, \mu, \nu, \rho)$ do *not* return to their original values; they are, rather, changed into their negatives. If, then, we regard the parametric four-dimensional world as *physically* significant, a rotation about a given direction through a given angle α is not *physically* the same as a rotation through an angle $\alpha + 2\pi$ about the same direction. In this kind of baseball one must run the bases twice before scoring! In more technical language the domain of 3×3 rotation matrices is *doubly covered.* If you are inclined to think lightly of these remarks as the mere speculations of a mathematician (which is just what they are), remember that they contain the whole idea of the spinning electron and of the calculus of spinors; it is a sobering thought that the "idea" underlying the spinning electron lay slumbering in a paper published by Euler in 1776 and that it had to wait 150 years before it was applied to physics.

2. Quaternions. The genius, or intuition, of Hamilton sensed that the light trying to shine through was largely dimmed out by the quadratic formulae implicit in the relation

$$O = E_3 + \sin\theta \cdot J + (1 - \cos\theta)J^2; \quad J^3 + J = 0.$$

If we regard J as in some way analogous to I the relation $J^3 + J = 0$ has lost the simplicity of the relation $I^2 + E_2 = 0$. How can this lost simplicity be regained? Hamilton divined that the proper thing to do is to pass on to four dimensions, and he was not deterred by the fact that, at first sight, the situation seems much worse. We write our alternating 4×4 matrix A in the form

$$A = \begin{matrix} 0 & -r & q & l \\ r & 0 & -p & m \\ q & p & 0 & n \\ l & -m & -n & 0 \end{matrix}$$

and observe that A satisfies the fourth degree equation

$$A^4 + (p^2 + q^2 + r^2 + l^2 + m^2 + n^2)A^2 + (pl + qm + rn)^2E_4 = 0.$$

It is convenient to regard A as identified by means of the *two* space vectors (p, q, r) and (l, m, n), but a word of caution concerning this identification is necessary. When a new reference frame is used, so that A is replaced by RAR', the two identifying vectors (p, q, r) and (l, m, n) will, in general, change. What remain unaffected by this change of reference frame are the coefficients of A^2 and E_4 in the fourth degree equation satisfied by A; in other words, the sum of the squared magnitudes of the two vectors and their squared scalar product are independent of the reference frame. Using the fourth degree equation satisfied by A, we may remove from e^A all powers of A above the third so that

$$O = e^A = c_0E_4 + c_1A + c_2A^2 + c_3A^3.$$

The values of the coefficients (c_0, c_1, c_2, c_3) may be readily found in the same manner as in the case of 3×3 matrices. There exists a rotation matrix R such that

$$RAR' = \begin{pmatrix} 0 & -\theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\phi \\ 0 & 0 & \phi & 0 \end{pmatrix}$$

where $\theta^2 + \phi^2 = p^2 + q^2 + r^2 + l^2 + m^2 + n^2$; $\theta^2\phi^2 = (pl + qm + rn)^2$. Then

$$e^{RAR'} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}$$

and we have merely to solve the equations

$$\cos\theta = c_0 - c_2 b \qquad \frac{\sin\theta}{\theta} = c_1 - c_3\theta^2$$

$$\cos\phi = c_0 - c_2\phi^2; \quad \frac{\sin\phi}{\phi} = c_1 - c_3\phi^2.$$

But this is the *wrong* road to follow. We should rather observe that whilst, *in general*, RAR' does *not* satisfy an equation of degree less than four it does satisfy an equation of degree *two* when $\phi^2 = \theta^2$. Since RAR' and A satisfy the same equation it follows that, when $\phi^2 = \theta^2$, A satisfies the second degree equation

$$A^2 + \theta^2 E_4 = 0.$$

Now $\phi^2 = \theta^2$ when, and only when,

$$(p^2 + q^2 + r^2 + l^2 + m^2 + n^2)^2 = 4(pl + qm + rn)^2$$

which implies either $l = p, m = q, n = r$ or $l = -p, m = -q, n = -r$. In other words, the 4×4 alternating matrix A satisfies a *second* degree equation when the two identifying vectors (p, q, r) and (l, m, n) are the same or opposite. We say that there are two classes of *special* alternating 4×4 matrices; the word special meaning that any matrix of either one of these two classes satisfies a second degree equation, whilst an alternating 4×4 matrix which does not belong to either of the two classes does not satisfy an equation of degree less than four. Each special 4×4 alternating matrix may be identified by a *single* vector (p, q, r). Let us denote any special matrix of the first class by a subscript 1; thus

$$A_1 = \begin{matrix} 0 & -r & q & p \\ r & 0 & -p & q \\ q & p & 0 & r \\ p & -q & -r & 0 \end{matrix}$$

and $A_1^2 + \theta^2 E_4 = 0$ where $\theta^2 = p^2 + q^2 + r^2$. Exactly as in the case of 2×2 matrices it follows that

$$O_1 = \cos\theta E_4 + \frac{\sin\theta}{\theta} A_1.$$

If we write A_1 in the form $pI_1 + qJ_1 + rK_1$ so that

$$I_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \quad J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

it is clear that each of the 4×4 alternating matrices I_1, J_1, K_1 is, as the subscript 1 implies, special and of the first class. For each of these three special matrices of the first class $\theta^2 = 1$ and so

$$I_1^2 + E_4 = 0; \quad J_1^2 + E_4 = 0; \quad K_1^2 + E_4 = 0.$$

On denoting by α, β, γ the unit vector along (p, q, r) we have $p = \theta\alpha$; $q = \theta\beta$; $r = \theta\gamma$ so that

$$O_1 = \cos\theta E_4 + \sin\theta(\alpha I_1 + \beta J_1 + \gamma K_1).$$

The analogy between this and the elementary de Moivre formula $O = \cos\theta E_2 + \sin\theta I$ is striking. The complex unit I is merely replaced by a *vector* complex unit $\alpha I_1 + \beta J_1 + \gamma K_1$. An easy calculation shows that

$$J_1K_1 = -K_1J_1 = I_1;\quad K_1I_1 = -I_1K_1 = J_1;\quad I_1J_1 = -J_1I_1 = K_1$$

and so $(\alpha I_1 + \beta J_1 + \gamma K_1)^2 = \alpha^2 I_1^2 + \beta^2 J_1^2 + \gamma^2 K_1^2 = -(\alpha^2 + \beta^2 + \gamma^2)E_4 = -E_4$. In other words the vector complex unit is, like i, a "square root of -1."

It is clear that if any linear combination of E_4, I_1, J_1, K_1 is orthogonal it must be an O_1. In fact the transpose of $aE_4 + bI_1 + cJ_1 + dK_1$ being $aE_4 - bI_1 - cJ_1 - dK_1$, the orthogonality of $aE_4 + bI_1 + cJ_1 + dK_1$ yields $(aE_4 + bI_1 + cJ_1 + dK_1)(aE_4 - bI_1 - cJ_1 - dK_1) = E_4$ and this implies $a^2 + b^2 + c^2 + d^2 = 1$, so that we may set $a = \cos\theta$, $b = \sin\theta\alpha$, $c = \sin\theta\beta$, $d = \sin\theta\gamma$ (where (α, β, γ) is a unit vector). It follows at once that the special 4×4 orthogonal matrices of the first class constitute a *group;* the product of any two elements of the class belongs also to the class.

The same remarks apply to the special 4×4 alternating matrices of the second class which we shall denote by the subscript 2. A typical element of this second class is

$$A_2 = \begin{pmatrix} 0 & -r & q & -p \\ r & 0 & -p & -q \\ -q & p & 0 & -r \\ p & q & r & 0 \end{pmatrix} = FA_1F$$

where $F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = F^{-1}$ is an orthogonal (but *not* a rotation) 4×4 matrix. Each $O_2 = e^{A_2}$ is of the form $\cos\theta E_4 + \sin\theta$ $(\alpha I_2 + \beta J_2 + \gamma K_2)$ where (α, β, γ) is a unit vector and

$$I_2 = FI_1F;\quad J_2 = FJ_1F;\quad K_2 = FK_1F.$$

Thus $I_2{}^2 + E_4 = 0$; $J_2{}^2 + E_4 = 0$; $K_2{}^2 + E_4 = 0$
$J_2K_2 = -K_2J_2 = I_2$; $K_2I_2 = -I_2K_2 = J_2$; $I_2J_2 = -J_2I_2 = K_2$.

It is clear from these remarks that the quest for an extension or

generalization of the imaginary unit i of complex numbers has been successful. We have, in fact, secured two vector complex units:

$$\alpha I_1 + \beta J_1 + \gamma K_1; \quad \alpha I_2 + \beta J_2 + \gamma K_2$$

where (α, β, γ) is any unit vector and we have in each case de Moivre's formula:

$$O_1 = e^{\theta(\alpha I_1 + \beta J_1 + \gamma K_1)} = \cos\theta E_4 + \sin\theta(\alpha I_1 + \beta J_1 + \gamma K_1)$$

$$O_2 = e^{\theta(\alpha I_2 + \beta J_2 + \gamma K_2)} = \cos\theta E_4 + \sin\theta(\alpha I_2 + \beta J_2 + \gamma K_2) = FO_1F.$$

The collection of 4×4 matrices O_1 constitutes a *subgroup* of the group of all 4×4 rotation matrices, as also does the collection of matrices $O_2 = FO_1F$; the only matrices common to these subgroups are the unit matrix E_4 and its negative $-E_4$ for each of which $\sin\theta = 0$. But the question remains: what about the 4×4 rotation matrices which belong to neither of the subgroups O_1 and O_2? To answer this question we first observe that *any* A_1 commutes with *any* A_2, the common value of A_1A_2 and A_2A_1 where

$$A_1 = \begin{pmatrix} 0 & -r & q & p \\ r & 0 & -p & q \\ q & p & 0 & r \\ p & -q & -r & 0 \end{pmatrix}; \quad A_2 = \begin{matrix} 0 & -n & m & -l \\ n & 0 & -l & -m \\ m & l & 0 & -n \\ l & m & n & 0 \end{matrix}$$

being

$$\begin{pmatrix} pl - qm - rn & pm + ql & pn + rl & rm - qn \\ pm + ql & qm - rn - pl & qn + rm & pn - rl \\ pn + rl & qn + rm & rn - pl - qm & ql - pm \\ rm - qn & pn - rl & ql - pm & pl + qm + rn \end{pmatrix}$$

It follows that any $O_1 = e^{A_1}$ commutes with any $O_2 = e^{A_2}$, the common value of O_1O_2 and O_2O_1 being $e^{A_1 + A_2}$. It is easy to see that any 4×4 alternating matrix may be written as the sum of two special 4×4 alternating matrices one of each class. In fact if $\tilde{A}$ denotes the matrix obtained from A by interchanging the two vectors (p, q, r) and (l, m, n) we have $A = A_1 + A_2$ where

$$A_1 = {}^1/_2(A + \tilde{A}); \quad A_2 = {}^1/_2(A - \tilde{A})$$

A_1 being a special matrix of the first class and A_2 a special matrix of the second class. Hence every 4×4 rotation matrix O may be factored thus:

$$O = O_1O_2 = O_2O_1$$

where O_1 is a special rotation matrix of the first class and O_2 is a special rotation matrix of the second class. We may say that the complete group of 4×4 rotation matrices is the *product* of the subgroup of *special* 4×4 rotation matrices O_1 by the subgroup of *special* 4×4 rotation matrices O_2.

It is now clear what a quaternion is and how natural a generalization of a complex number it is. Instead of a single imaginary unit i we have three imaginary units i, j, k and the typical quaternion is $xi + yj + zk + t$ just as the typical complex number is $x + yi$. For complex numbers we have a single natural realization

$$x + yi \rightarrow xE_2 + yI = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

whilst when dealing with quaternions we have two natural realizations:

1. $xi + yj + zk + t \rightarrow xI_1 + yJ_1 + zK_1 + tE_4 =$

$$\begin{pmatrix} t & -z & y & x \\ z & t & -x & y \\ -y & x & t & z \\ -x & -y & -z & t \end{pmatrix}$$

2. $xi + yj + zk + t \rightarrow xI_2 + yJ_2 + zK_2 + tE_4 =$

$$\begin{matrix} t & -z & y & -x \\ z & t & -x & -y \\ -y & x & t & -z \\ x & y & z & t \end{matrix}$$

In complex number theory we have associated with the complex number $z = x + yi$ the plane vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ and if O is any 2×2 rotation matrix (so that $O = \cos\theta E_2 + \sin\theta I$) the vector Ov is associated with the product $(\cos\theta + \sin\theta i)z$. In quaternion theory we have associated with the quaternion $\xi = xi + yj + zk + t$ the four-dimensional vector $v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ and if O_1 is any *special* 4×4 rotation matrix of the *first* class (so that $O_1 = \cos\theta E_4 + \sin\theta(\alpha I_1 + \beta J_1 + \gamma K_1)$ where (α, β, γ) is any unit three-dimensional vector) the vector O_1v is associated with the product $q\xi$ where q is the unit quaternion $q = \sin\theta(\alpha i + \beta j + \gamma k) + \cos\theta$. On the other hand, if O_2 is any *special* 4×4 rota-

tion matrix of the *second* class (so that $O_2 = \cos\theta E_4 + \sin\theta(\alpha I_2 + \beta J_2 + \gamma K_2)$ where (α, β, γ) is any unit three-dimensional vector) the vector $O_2 v$ is associated with the product ξq^{-1} where $q^{-1} = -\sin\theta(\alpha i + \beta j + \gamma k) + \cos\theta$ is the unit quaternion reciprocal to q. Since quaternion multiplication is not commutative, we must pay strict attention to the order of the factors; thus when we operate on v by an O_2 our multiplying quaternion q^{-1} must appear on the right. It is also important to notice that it is q^{-1} and *not* q. If O is *any* 4×4 rotation matrix, it follows from the factorization $O = O_1O_2 = O_2O_1$ that if q_1 and q_2 are the unit quaternions that are realized by O_1 and O_2, respectively, then the vector Ov is associated with the quaternion $q_1\xi q_2{}^{-1}$.

What is the situation as regards 3×3 rotation matrices with which Hamilton, in the first instance, concerned himself? To answer this question we observe that when the vector (l, m, n) is the zero vector the corresponding special 4×4 alternating matrix A is such that $O = e^A = \begin{pmatrix} O_3 & 0 \\ 0 & 1 \end{pmatrix}$ where O_3 is a 3×3 rotation matrix. In this case

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & r \\ -p & -q & -r & 0 \end{pmatrix}$$

so that $A_1 = {}^1/_2\begin{pmatrix} 0 & -r & q & p \\ r & 0 & -p & q \\ -q & p & 0 & r \\ -p & -q & -r & 0 \end{pmatrix}$; $A_2 = FA_1F$. Hence $O_1 = e^{A_1}$ and $O_2 = e^{A_2}$ realize the *same* quaternion $q = \sin\frac{\theta}{2}(\alpha i + \beta j + \gamma k) + \cos\frac{\theta}{2}$ where $\theta = (p^2 + q^2 + r^2)^{1/2}$ and $p = \theta\alpha$, $q = \theta\beta$, $r = \theta\gamma$; if v is the three-dimensional vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ associated with the quaternion $\xi = xi + yj + zk$ the three-dimensional vector O_3v is associated with the quaternion $q\xi q^{-1}$. It is, however, clear that the realization is properly *four* dimensional rather than three dimensional; for O_3 must be factored into two four-dimensional special rotation matrices, and to do this it must be *extended* to an O_4 thus:

$$O_4 = \begin{matrix} O_3 & 0 \\ 0 & \end{matrix}$$

3. The application of quaternions to the relativistic wave equation. The theory of quaternions is admirably suited to a discussion of the relativistic wave equation proposed by Dirac. We shall content ourselves here with merely indicating this application of quaternions and shall refer the interested reader to the paper by A. W. Conway, "Quaternion Treatment of the Relativistic Wave Equation," *Proc. Roy. Soc.*, A, v. 162 (1937), pp. 145–154.

We first consider an electron of mass m in free space (so that the electromagnetic field acting on it is zero). Its wave function ψ is a four-dimensional vector $\begin{pmatrix}\psi_x\\ \psi_y\\ \psi_z\\ \psi_t\end{pmatrix}$ and we denote by ξ the quaternion $\psi_x i + \psi_y j + \psi_z k + \psi_t$ associated with ψ. Each component of ψ must satisfy the wave equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{h^2}\right) f = 0,$ where c is the velocity of light and h is Planck's constant. If we denote by Δ the quaternion operator:

$$\Delta = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

it is clear that the operators Δ and $\frac{\partial}{\partial t}$ are commutative. Since i and j are anti-commutative (i. e., $ji = -ij$) it follows that the two operators $\Delta\xi i$ and $\frac{1}{c}\frac{\partial}{\partial t}\xi j$ are anti-commutative; we here understand by $(\Delta\xi i)\left(\frac{1}{c}\frac{\partial}{\partial t}\xi j\right)$ the operator $\Delta\frac{1}{c}\frac{\partial}{\partial t}\xi ji$ and by $\left(\frac{1}{c}\frac{\partial}{\partial t}\xi j\right)(\Delta\xi i)$ the operator $\frac{1}{c}\frac{\partial}{\partial t}\Delta\xi ij$. In fact any two of the three operators $\Delta\xi i$, $\frac{1}{c}\frac{\partial}{\partial t}\xi j$, ξk, are anti-commutative. Since $\Delta^2 = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ it follows that the square of the operator $\Delta\xi i + \frac{1}{c}\frac{\partial}{\partial t} ej + \frac{mc}{h}\xi k$ is $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{h^2}\right)\xi$. Thus every quaternion solution ξ of the equation

$$\Delta\xi i + \frac{1}{c}\frac{\partial}{\partial t}\xi j + \frac{mc}{h}\xi k = 0$$

has associated with it a four-dimensional wave function ψ which satis-

fies the equation appropriate to an electron free from any influencing electromagnetic field. The equation

$$\Delta \xi i + \frac{1}{c}\frac{\partial}{\partial t}\xi j + \frac{mc}{h}\xi k = 0$$

is Dirac's equation for a free electron. When there is an influencing electromagnetic field, this field is most conveniently specified by its scalar potential ϕ and its vector potential (A_x, A_y, A_z). Denote by π the quaternion $A_x i + A_y j + A_z k$ and by ι the ordinary imaginary unit; then the wave equation for an electron in the presence of the electromagnetic field is obtained from the wave equation of the *free* electron by merely replacing Δ by $\Delta + \frac{\iota e}{ch}\pi$ and $\frac{1}{c}\frac{\partial}{\partial t}$ by $\frac{1}{c}\frac{\partial}{\partial t} - \frac{\iota e}{ch}\varphi$ where e is the charge on the electron.

THE JOHNS HOPKINS UNIVERSITY

POEMS

By William Rowan Hamilton

O brooding Spirit of Wisdom and of Love,
Whose mighty wings even now o'ershadow me,
Absorb me in thine own immensity,
And raise me far my finite self above.
Purge vanity away, and the weak care
That name or fame of me may widely spread.
And the deep wick keep burning in their stead,
Thy blissful influence afar to bear—
Or see it borne. Let no desire of ease
No lack of courage, faith, or love, delay
Mine own steps, or that high thought-paven way
In which my soul her dear commission sees.
Yet with an equal joy let me behold
Thy chariot o'er that way by others rolled.

* *
*

Who says that Shakespeare did not know his lot,
But deem'd that in time's manifold decay
His memory should die or pass away,
And that within the shrine of human thought
To him no altar should be reared? O hush!
O veil thyself awhile in solemn awe!
Nor dream that all man's mighty spirit-law
Thou know'st; how all the hidden fountains gush
Of the soul's silent prophesying power,
For as deep love, mid all its wayward pain,
Cannot believe but it is loved again
Even so, strong genius, with its ample dower
Of a world-grasping love, from that deep feeling
Wins of its own wide sway the clear revealing.

HAMILTON'S WORK IN DYNAMICS AND ITS INFLUENCE ON MODERN THOUGHT

By H. Bateman

INTRODUCTION

IN his admirable report on recent progress in theoretical dynamics, Cayley[1] says that Hamilton's memoirs[2, 3] of 1834 and 1835 began a second period in the history of the subject. Influenced no doubt by what had already been accomplished in his earlier work on systems of rays Hamilton tried to find a function by means of which the solutions of the dynamical equations could be derived in a simple way by partial differentiation with respect to a number of parameters. As a result of this effort he made three important advances and it is customary now to speak of Hamilton's principle, Hamilton's canonical equations of motion, and the Hamilton-Jacobi partial differential equation.

HAMILTON'S PRINCIPLE

This was a formulation of a principle of least action, not unlike Fermat's principle of least time, from which analytical dynamics could be derived by the application of Lagrange's methods of the calculus of variations. Hamilton recognized the great generality of this method in his two memoirs and in a later note[3] he applied it to one of those general variation problems which are now associated with the name of A. Mayer. The method was soon applied to the mechanics of continuous media by Green,[4] Clebsch,[5] Thomson and Tait[6] while applications to electrodynamics were considered by F. E. Neumann[7] and Clausius.[8] It then seemed likely that the method would be applicable to the whole of physics and this view was advocated by Larmor[9] and Helmholtz[10] in important papers. Meanwhile notes on the history of the subject were written by Mayer,[11] Larmor,[12] and Helmholtz.[13] A firm belief in the generality of Hamilton's principle has been a guiding idea in many of the great developments of mathematical physics in recent years. Variational principles of the Hamiltonian

type were used by Mie[14] in his theory of matter, by Hilbert,[15] Einstein,[16] Lorentz,[17], Tresling,[18] Fokker,[19] Weyl,[20] Palatini,[21] Whittaker,[22] and others in the development of the general theory of relativity. De Broglie[23] and Schrödinger[24] used them in the development of the new quantum theory and when Dirac[25] began to develop his modification of quantum theory and ideas about matter Darwin[26] showed that by combining atom and radiation field into a single system the whole scheme of equations could be derived from a variational principle.

IS HAMILTON'S PRINCIPLE ALWAYS APPLICABLE?

The universality of Hamilton's principle and of the principle of least action has been contested. Maupertuis[27] was apparently under the impression that his principle of least action was more general than the principle of *vis viva* but for the validity of his principle the constancy of the energy must be assumed. Jacobi[28] mentioned this point at the beginning of his sixth lecture on dynamics. Hamilton's principle is usually enunciated for conservative holonomic systems as in Whittaker's *Analytical Dynamics*.[29] The kinetic potential L is then $T - V$, where T is the kinetic energy and V the potential energy. When L does contain the time explicitly the principle states that Ldt has a stationary value for any part of an actual trajectory as compared with neighboring paths which have the same terminal points and associated times as the actual trajectory. When L does not contain the time explicitly the last condition may be replaced by the condition that the total time of description is to have the same value in the two cases.

Hertz,[30] who introduced the term non-holonomic systems, held the opinion that Hamilton's principle is valid for holonomic systems alone. According to Jourdain,[31] Routh communicated to Ferrers the fact that Lagrange's equations of motion fail for non-holonomic systems and Ferrers[32] mentioned this failure in a paper. Routh[33] then gave a form of the equations of motion which is valid for such systems. If the non-integrable equations of connection which characterize the non-holonomic system are

$$L_s \equiv A_{s1}\, q_1' + A_{s2}\, q_2' + A_{sn}\, q_n' = 0 \qquad (s = 1, 2, \ldots m)$$

Routh's equations are

$$d/dt(\partial L/\partial q_r') - \partial L/\partial q_r = \lambda_1 \partial L_1/\partial q_r' + \lambda_2\, \partial L_2/\partial q_r' + \ldots$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are parameters whose values are to be found. Jourdain gives two other forms of the equations of motion for a non-holonomic system and mentions that there had been an impression that the

difficulties relating to rolling motion, a particular non-holonomic system, were first pointed out by C. Neumann.[34]

Jourdain also points out that when the differential equations of motion are derived from Hamilton's principle or from the principle of least action as used by Maupertuis and Euler, they take Routh's form. The types of variation developed by Hölder[35] for these principles make them equivalent in all respects to the principle of d'Alembert without even the supposition of the existence of a force-function, the holonomic character of the connections or the independence of the equations of condition of the time t. Kerner[36] and Hamel[37] have discussed the matter further. In the classical sense the virtual displacements which are to be made in accordance with d'Alembert's principle are not permissible. Kerner considers the system of partial differential equations obtained by equating the expressions in the actual equations of motion to the expressions derived from the variational principle by the rules of Euler and Lagrange while Hamel points out that Hamilton's principle fails only when complete equivalence of the variational equations and the equations of motion is demanded. The right equations of motion are obtained eventually by a suitable choice of the parameters λ_r.

Bilimovitch[38] has paid special attention to conservative non-holonomic systems in which the coefficients in the non-integrable linear conditions depend on the time. These systems have also been discussed by Pöschl.[39] Delassus[40] has developed a theory in which the connections of the system are not necessarily linear and Béghin[41] indicates that the ordinary theories are insufficient for the cases of communicating systems such as those occurring in the theory of the gyrostatic compass of Anschütz and Sperry. This matter is discussed further by Przeborski.[42]

Schieldrop[43] introduced a quadratic differential form D to represent what he called the deviation and thus gave a kinematic interpretation of the terms in a non-holonomic system which supplement the usual Lagrangian terms. Johnsen[44] has generalized it.

Taylor[45] and Campbell[46] have discussed the characterization of holonomic or non-holonomic systems by means of integral invariants.

For further discussion of non-holonomic systems reference may be made to standard works such as those of Kirchhoff[47] and Appell[48] and Kirchhoff's presentation is examined critically by Schaefer[49] who offers a simplified treatment.

The case of non-conservative systems is discussed by Garcia[50] who uses both classical and relativistic mechanics, the method of treatment being similar to that of Levi-Civita.[51]

Guillaume[52] maintains that Hamilton's principle applies rather badly to Maxwell's electromagnetic theory and to electron theory and recommends in its place the general equations considered by Appell.[53] This author writes the equations of motion in the form

$$\partial U/\partial q_r = \partial A/\partial q_r''$$

where A denotes the summation $\frac{1}{2}\sum_r m_r(x''^2 + y''^2 + z''^2)$ and U is the force function. These equations, as Jourdain[54] shows, can be derived by the methods of the Calculus of Variations from a principle of type

$$\delta(A - U'') = 0$$

where it is understood that $\delta U = \sum_v (X_v\delta x'' + Y_v\delta y'' + Z_v\delta z'')$, x_v, Y_v, Z_v being component forces. Jourdain shows also that this variational principle is equivalent to Gauss's principle of Least Constraint. Problems of electrodynamics in which there is radiation present the difficulty that the system is dissipative and Legendre's conditions for a maximum or minimum are not satisfied, the partial differential equations being of hyperbolic type. The same difficulty occurs in the supersonic flow of a fluid and there is a similar difficulty when there is diffusion of some kind as in a viscous fluid in laminar or turbulent motion. There is still some difference of opinion regarding the applicability of variational principles to dissipative systems. Jeffery[55] considered motions of solid particles in a viscous fluid which would give a minimum dissipation of energy. Millikan[56] discussed the possibility of deriving just the equations of motion of an incompressible fluid and the equation of continuity from a variational principle of the usual type and concluded that it was impossible. Dissipative systems were discussed by Bauer[57] with the same general result so long as the desired equations were required to be obtained directly in a specified form. The present author[58] showed, however, that an equivalent form was derivable from a variational principle and took the view that the apparent failure of Hamilton's principle was in many cases due to the fact that the specified dynamical system was incomplete. If another system were added to absorb the energy so that the whole system became conservative, the equations of motion of the whole system should be derivable from Hamilton's principle. It is always possible to derive a given set of equations and certain other equations connecting a different set of dependent variables with the given ones from a suitable variational principle.

Comments on the two views were made by Synge[59] who concluded that in the case of the two equations of Whittaker cited by the present

author, there was no variational principle which would give the trajectories of the system as the sole extremals. Synge also tried to discuss the conditions under which the equations of motion of a dissipative system could be derived from a variational principle.

The present author[60] upheld his point of view when Whittaker[61] stated in a review that the equation of the conduction of heat can be regarded only as a limiting case of an equation arising in the calculus of variations. The example given to refute this view is as follows: Let $u = F(x, t)$, $v = F(x, -t)$ and let the integration be over a symmetric domain so that the variational principle is of type

$$\int_{-a}^{a} \int_{-b}^{b} (u_x v_x + v u_t) dx dt = 0$$

where suffixes are used to denote partial derivatives. The equations given by the usual methods are then

$$u_{xx} = u_t, \quad v_{xx} = -v_t$$

and these are equivalent to just one equation for the function $F(x, t)$. This, of course, is a very special system which is hard to interpret physically. Generally a second system to absorb energy could be added to a given dissipative system in an infinite number of ways and it may be hard to select one of physical interest. This holds also in cases where there is a diffusion of particles.

In spite of the difference of opinion regarding dissipative systems Hamilton's principle is still being used as a basis for the creation of unified theories of electricity and gravitation. Reference may be made, for instance, to Schrödinger's attempt[62] to derive the meson field from a variational principle which also gives the gravitational and electromagnetic fields and to Einstein's most recent unified field theory.[63]

Modifications of Hamilton's principle which will give the right equations of motion have been considered. Thus Holzmüller[64] points out that for mechanical problems in which the force function U contains the coordinates, the time, and the velocities the principle of Least Action in the form

$$\delta \int_{t_0}^{t_1} (T + U) dt = 0$$

does not generally give the right equations of motion but when U is replaced by a suitable function U' the correct equations are obtained. He considers in particular the electrodynamical law of Weber for which the force function is

$$U = (m/r)\,(1 - r^{2\prime}/c^2)$$

and finds that correct results may be obtained by using in place of U Neumann's function

$$U' = (m/r)\,(1 + r^{2\prime}/c^2).$$

For further discussion of variational principles for mechanics reference may be made to two papers by Lipschitz[65] in which generalized coordinates are used and to the encyclopedia articles of Voss[66] and Prange.[67]

HAMILTON'S CANONICAL EQUATIONS AND THE HAMILTON-JACOBI PARTIAL DIFFERENTIAL EQUATION

If L is the kinetic potential for a conservative holonomic system described by means of the generalized coordinates q_r, the expression

$$H = \sum_{r=1}^{n} p_r q_r' - L, \text{ with } p_r = \partial L/\partial q_r'$$

is such that

$$\delta H = \sum^{n} (q_r'\,\delta p_r - p_r'\,\delta q_r)$$

and so

$$dq_r/dt = \partial H/\partial p_r,\ dp_r/dt = -\partial H/\partial q_r \qquad (r = 1, 2 \ldots n).$$

These are called the canonical equations. When L and consequently H do not contain t explicitly the energy equation is simply

$$H(q_1, q_2, \ldots q_n, p_1, p_2, \ldots, p_n) = h$$

where h is a constant. In the report already mentioned and in a later report Cayley[68] remarks that Cauchy had mentioned these canonical equations in a memoir read in 1831 but not published so Hamilton was unaware of the contents. Cayley also mentions two papers of Ostragradsky, published in 1848 and 1850 in which it is shown that when the force function contains the time, the equations of motion can be transformed from the Lagrangian to the Hamiltonian form. Ostragradsky showed also that the general isoperimetric problem of the Calculus of Variations can be transformed to the Hamiltonian form.

When the canonical equations are written in the form

$$dq_1/\partial H/\partial p_1 = dq_2/\partial H/\partial p_2 = \ldots = dp_1/(-\,\partial H/\partial q_1) = \ldots = dt$$

they resemble the equations of Lagrange and Charpit associated

with a partial differential equation and in the theory of Cauchy the canonical equations become the differential equations of the characteristics of this partial differential equation. Hamilton had the insight to see that it was worth while to proceed in the other direction. Instead of regarding the canonical equations as an aid in the solution of the partial differential equation he proposed to use the partial differential equation in the same way as the equation for the characteristic function in optics and to derive the solutions of the equation of motion from a complete integral of the partial differential equation. He actually constructed two partial differential equations, using also a second principal function H' defined by the equation

$$H' = \sum_{r=1} (p_r' q_r) + L.$$

The idea of using a partial differential equation was adopted eagerly by Jacobi[69] and applied to many problems. For twenty years or more mathematicians were indeed busy applying Hamilton's methods to special problems. Brioschi,[70] in particular, treated the motion of a particle on a surface and compared Hamilton's method with the method used by Bertrand. In the special case when the surface is a sphere the problem is identical with that of the motion of a spherical pendulum.

The canonical equations were well adapted for the study of the transformation of dynamical problems. Bertrand[71] and Donkin[72] were early investigators in this subject but the greatest developments were made by Sophus Lie[73] whose philosophy is well described by a passage quoted by Lovett[74] from which we shall make the following translated extract. "How should we represent natural phenomena if not by a succession of transformations of which the laws of the universe are the invariants. In developing his philosphy Lie paid special attention to contact transformations and the transformation theory of dynamics became associated with the general theory of the covariance of the equations of the characteristics of a partial differential equation when use is made of a contact transformation. It would occupy too much space to give a full discussion of the transformation of dynamical problems. Reference for this may be made to the books of Whittaker,[75] Birkhoff[76] and Wintner;[77] papers by Appell,[78] Vergne,[79] Wright,[80] Birkhoff,[76] Lewis,[81] van Kampen and Wintner.[82] One way of deriving transformations is to make the difference of two Pfaffians (in different variables) an exact differential. In addition to the transformations which can be applied to an arbitrary canonical system (the contact transformations) there are other transformations of a more general

nature which can be applied to particular dynamical systems. These transformations have been discussed by Bilimovitch[83] who gives equations by means of which such transformations can be discovered.

These transformations may be compared with the transformations of potential theory and electromagnetic theory in which the transformation and field are related. These transformations have been studied to some extent by the present author.[84] As an example of such a transformation we may mention the search for solutions of Laplace's equation of the form

$$V = wF(u, v)$$

where u, v, w are special functions of x, y and z while F is any solution of a partial differential equation. In the special case in which

$$u = x/(z + r),\ v = y/(z + r),\ r^2 = x^2 + y^2 + z^2$$

there are many forms of w that can be used. In particular, if

$$w = r^n$$

the equation for F is

$$4n(n + 1)\ (u^2 + v^2 + 1)^2\ F + F_{uu} + F_{vv} = 0.$$

If Laplace's equation is replaced by the wave-equation

$$W_{xx} + W_{yy} = (1/c^2)W_{tt}$$

there is a solution of type $W = wf(u, v)$ corresponding to that of Laplace's equation where now

$$u = x/(ct + s),\ v = y/(ct + s),\ s^2 = c^2t^2 - x^2 - y^2.$$

If $W = s^n$ the equation for f is easily found. This is just one case in which a varying system can be transformed into a static system. Such transformations are very numerous and are of some philosophic interest because there is the possibility that the mind's mode of interpretation of the processes taking place in the brain becomes adjusted to them in such a way that the normal functioning of the body is obscured and attention can be paid at low power to the small departures from the norm.

The solutions of Laplace's equation of form $wF(u, v)$ include as a a special case the Lame products of type $A(a)B(b)C(c)$ which occur when the transformed equation in the coordinates a, b, c is separable. This corresponds to the dynamical problem of finding cases like those of Liouville[85] and Stäckel[86] in which the dynamical equations have solutions of a particularly simple type in which a number of functions,

each of only one variable, occur. These solutions are of much interest in quantum theory and have been studied in this country by Robertson[87], and Eisenhart.[88] There have been many applications of Hamilton's canonical equations and partial differential equation in quantum theory.

In Bohr's atomic theory as well as in Celestial Mechanics the conditionally periodic motions played an important part. In certain cases the solution of the canonical equations for an analytic function H can be expressed explicitly in the form

$$p_r = P_r\,(c_1, c_2, \ldots, c_n;\ w_1, w_2, \ldots, w_n)$$

$$q_r = Q_r\,(c_1, c_2, \ldots, c_n;\ w_1, w_2, \ldots, w_n)$$

where the quantities w, are angular variables of type $a_r t + b$, with coefficients a_r depending on the quantities c_r. Levi-Civita[89] remarked that the opinion has been expressed that any mechanical system has such a solution but with the aid of Weierstrass's preparation theorem he tried to show that the solubility by angular coordinates implies the existence of n uniform integral relations.

In the formulation of quantum conditions Hamilton's methods have been used with suitable modifications. In the many electron problems in which Pauli's exclusion principle must be used certain non-commutative rules of multiplication are added. In the presentation of the principle by Jordan and Wigner, Dirac's approximate formula for the mutual action of an arbitrary number of equivalent particles endowed with spin is simply derived. In this connection reference may be made to papers of Heisenberg[90] and Jordan.[91]

In the method of Weierstrass and Hilbert[92] in the Calculus of Variations use is made of an invariant integral which for the problem

$$\delta \int f(y, y')dx = 0$$

has the form

$$I^* = \int \left\{ f(y, p)dx + (dy - pdx)f_{y'}\,(y, p)\right\}.$$

There is an associated partial differential equation satisfied by a principal function which in the case of the problem of shortest distance reduces to Hamilton's partial differential equation for his principle function.

In the quantization of field equations, Born [93] proposed to use Hilbert's independence theorem as a possible basis for quantization. Weyl[94] criticized this proposal and concluded that it is too narrow. Dirac[95] has indicated some cases in which dynamical systems are ex-

ceptional because the Lagrangian and Hamiltonian methods cannot be used directly. The relativistic motion of a particle whose proper mass is zero is cited as an example. Dirac shows, however, that the difficulties in the use of Hamiltonian methods may be overcome by the introduction of homogeneous variables. In the study of the stability of dynamical systems it is often advantageous to start with equations of motion which differ only slightly from a specified set of canonical equations with periodic solutions or other solutions of a known type. The work of Birkhoff and Lewis[96] in this country and of Persidskij[97] and Pontrjagin[98] in Russia may be cited in illustration. The method is something like the method of perturbations of celestial mechanics.

The reduction of dynamics to geometry soon followed the development of differential geometry and the use of such geometrical ideas by Hamilton, Jacobi, and Liouville gave a great impulse to this mode of representation. According to Stäckel[99] two dynamical problems are analytically equivalent when both belong to the same problem of the motion of a point in an n-fold manifold. This idea was generalized by Appell.[100] The geometrical aspect of dynamics has been discussed in a general way by Lewis.[101]

There are still greater generalizations than those considered by Stäckel and Appell. Hamilton's ideas have been used in connection with functional and abstract spaces by Lanczos,[102] Birkhoff,[103] and Michal.[104] Use has been made of transformations in Hilbert space and singular integral equations by Koopman[105] and Carleman[106] for the treatment of Hamiltonian systems and the differential equations of general dynamics.

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Hamilton's Contribution To Mechanics*

By Edwin B. Wilson

Perhaps an even more important contribution to science by Hamilton than his quaternions is in his work on mechanics. Hamilton's Principle and Hamilton's Equations have found great developments by other scientists and I understand that it was to Hamilton that Schroedinger went back in developing his wave mechanics. By and large there is no such history of subsequent developments in the line of Hamilton's quaternions in pure mathematics. It is true that Benjamin Peirce and his son Charles Peirce made some generalizations of a sort in their work on linear associative algebra but they did so, I think, by departing very greatly from the point of view of Hamilton himself.

* Excerpt from a letter to the editor.

THE CONSTANCY OF THE VELOCITY OF LIGHT

THE INVARIANCE OF MAXWELL'S ELECTROMAGNETIC FIELD EQUATIONS UNDER RELATIVITY TRANSFORMATION, DEMONSTRATED IN FOUR DIMENSIONS*

BY VLADIMIR KARAPETOFF†

1. The Purpose of This Article

ONE of the fundamental postulates of Einstein's theory of relativity is that the velocity of light in vacuo is the same for all observers, independent of their motion with respect to each other. It is the purpose of this article to show that this constancy of the velocity of light may be deduced directly from certain differential properties of a representative four-dimensional space. An elementary proof of this invariance in a three-dimensional space is tedious and perhaps not quite rigorous (Ref. 1, p. 152).

Without any reference to relativity, the velocity of light in vacuo may be deduced directly from Maxwell's differential equations of electromagnetic field, in terms of the electric permittivity and magnetic permeability of the vacuum. This deduction may be found in many advanced books on electromagnetism, electromagnetic field, and propagation of electromagnetic waves. Since, in relativity, the velocity of light is to remain unchanged when an observer is moving towards or away from the source of a beam of light, the only logical conclusion is that the Maxwell equations are invariant to a change from stationary to moving axes of coördinates.

However, merely changing the distance and time coördinates in accordance with the principles of relativity, is not sufficient. As is shown at the end of Reference 1, arbitrary assumptions had to be made in regard to the transformations of the components of the electric intensity **E** and magnetic intensity **H** to preserve the invariance of these four equations. By assuming **E** and **H** to be two mutually perpendicular linear vectors in the usual sense, normal to the direction of the

* Abridged from the author's forthcoming book on relativity.

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propagation of the wave, it does not seem possible to demonstrate the invariance of Maxwell's equations between the two observers. On the other hand, by resorting to a four-dimensional quasi-Euclidean space, with the fourth dimension being used for plotting time intervals as imaginary quantities, and thinking of **E** and **H** as being associated with directed elements of area, the desired invariance of Maxwell's equations may be shown to hold true (Ref. 2, Chap. 5).

The author's previous work in relativity having been largely confined to hyperbolic coördinates, it has seemed natural to attempt a representation of the Maxwell equations of electromagnetic field in a semi-hyperbolic representative space, with a real time axis, the change from one observer to the other being accomplished by a rotation of the hyperbolic XT-plane by a hyperbolic angle, without disturbing the vectorial construction of the electromagnetic field itself in the four-dimensional space.

It is shown below that a small directed area K may be imagined at any point within this representative space to determine the electric and magnetic intensities at the point under consideration. This directed area will be referred to as a *plane* or *area vector* in contradistinction to the usual linear vectors. The divergence and the curl of either a plane or a linear vector in the four-dimensional space will be referred to as the four-divergence and the four-curl, respectively. The four-curl and the four-divergence of K are then shown to be analytical expressions of the four Maxwell equations. These two operators being intrinsically invariant to a rotation of the XT-plane, the Maxwell equations of necessity preserve their form between the two observers. Thus, the invariance of these equations is demonstrated to be a direct result of certain differential properties of the suitable representative space, rather than one of specific physical assumptions.

It is hoped that this development may be of interest to pure mathematicians concerned with the vector analysis of a four-dimensional space as well as to physicists who use the invariance of Maxwell's equations either without an adequate proof or as a result of a tedious transformation of several partial derivatives.

2. Maxwell's Electromagnetic Field Equations[1].

These equations, in the Gaussian system of units, and in the language of the three-dimensional vector analysis, are as follows:

[1] For a deduction of Maxwell's equations and their solution of the velocity of light, c, see a modern text on electromagnetic theory, field, or waves.

$$\nabla\cdot\mathbf{H} = 0;\ \nabla\times\mathbf{E} + \frac{\partial\mathbf{H}}{\partial(ct)} = 0 \tag{1}$$

$$\nabla\cdot\mathbf{E} = \rho;\ \nabla\times\mathbf{H} - \frac{\partial\mathbf{E}}{\partial(ct)} = \mathbf{j} \tag{2}$$

These four equations refer to a point in the field at which $\mathbf{H}$ is the vector of the magnetic field intensity, $\mathbf{E}$ is the vector of the electrostatic field intensity, ρ is the density of the electrostatic charge, and $\mathbf{j}$ is the vector of the current density. The letter t denotes variable time and c is the velocity of light in vacuo. $\mathbf{H}$ and $\mathbf{j}$ are expressed in electromagnetic units, whereas $\mathbf{E}$ and ρ are in electrostatic units. With these assumptions, the physical dimensions of $\mathbf{H}$ and $\mathbf{E}$ are the same, and so are those of $\mathbf{j}$ and ρ. This circumstance makes the use of the Gaussian system of units particularly convenient in the subsequent graphical treatment in a four-dimensional representative space.

The equation $\nabla\cdot\mathbf{H} = 0$ signifies the well-known physical fact that the magnetic flux is solenoidal in its very nature; in other words, its divergence is equal to zero at any point in the field. The second equation (1) expresses Faraday's law of induced electromotive force; $\mathbf{H}$ is the magnetic flux per unit area and $\nabla\times\mathbf{E}$ is the electromotive force induced in the perimeter of this area and equal to the rate of change of $\mathbf{H}$ with the time. The factor c is necessary because $\mathbf{H}$ is expressed in electromagnetic units whereas $\mathbf{E}$ is in the corresponding electrostatic units.

The equation $\nabla\cdot\mathbf{E} = \rho$ means that the divergence of the electrostatic flux is numerically equal to the density of electric charge at the point under consideration. This is the so-called Gauss's Theorem expressed in the language of vector analysis. The second equation (2) is the so-called law of circuitation applied to the magnetomotive force and the flux at a point in the field. The expression $\mathbf{j} + \partial\mathbf{E}/\partial(ct)$ represents the electric excitation flowing through a unit area and $\nabla\times\mathbf{H}$ is the resulting flux density over the perimeter of this area. In the most general case, this magnetomotive force consists of an actual current, $\mathbf{j}$, and of a displacement of electric charges whose rate of motion with the time, in vacuo, is $\partial\mathbf{E}/\partial(ct)$. The velocity of light, c, again enters because $\mathbf{H}$ and $\mathbf{j}$ are in electromagnetic units, whereas $\mathbf{E}$ is in the corresponding electrostatic units.

The foregoing four equations are assumed to hold true for a particular observer whom we shall call S, and the purpose of this article is to show that identical equations hold true for another observer, S′, who is in uniform relative motion with respect to S. Of course, primed quantities will enter in the expressions for the S′ observer, being

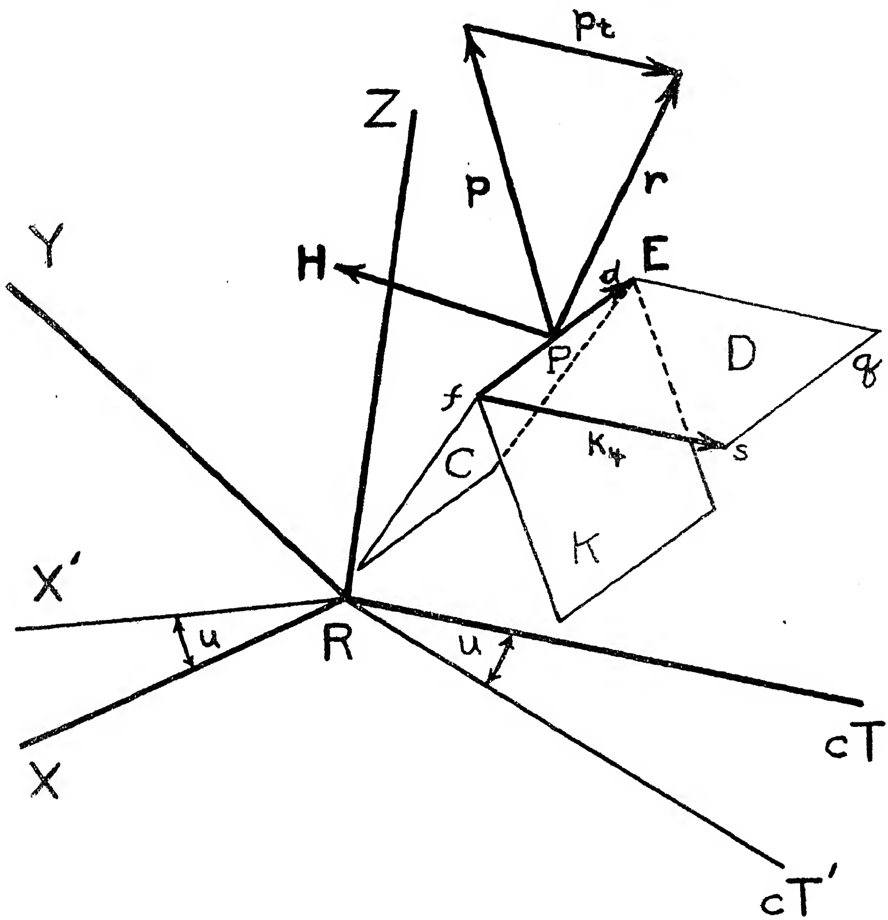
the values actually measured by him, but the form of the equations remains the same so that the same value of c is computed from either set, which agrees with the fundamental postulate of the theory of relativity.

3. The Field Plane K and Its Components, C and D, in the Four-Dimensional Representative Space.

We shall now deduce some invariant properties of a curl and a divergence of a vector in a representative four-dimensional space and then show that the foregoing equations (1) and (2) are but special cases of these invariants in application to a physical problem. The reader is familiar with the curl and the divergence of a three-dimensional linear vector. Here, instead of a linear vector we shall deal with an area vector which in a continuous field of force has definite magnitude and direction at every point of the field. To make this concept clearer, imagine a three-dimensional field in which the velocity of motion, the electric force, or the temperature at every point is represented by a line of definite magnitude and direction. Now, in our four-dimensional space we shall imagine at every point a small area of definite magnitude and direction, which area may represent a directed physical quantity at that point or for the present be only a mathematical concept. The field of force under consideration is still a vector field, only the vector at the point is now an area vector and not a linear vector. For the purpose of this article, the shape of the area is of no consequence, but only its magnitude and direction in space. In the next two sections the curl and the divergence of such an area in the four-dimensional space will be defined and their expressions deduced. We shall first give a schematic representation in perspective of such a field area with its components and some other quantities which enter in the problem.

In the diagram, R is the origin of a set of coördinates RX, RY, RZ, RcT, in the chosen four-dimensional representative space. The axes RX, RY, RZ are orthogonal, but the axis RcT is hyperbolic with respect to the RX-axis. Since distances are plotted along the first three axes, it is convenient to plot values of ct along the fourth axis, rather than those of time, t. The quantity c being the constant velocity of light, it merely changes the scale along the fourth axis. However, ct is a distance and not an interval of time, so that in this manner distances are plotted along all the four axes and the representative space becomes homogeneous, which is indispensable in the construction used below. The above axes are those of the S observer, and the construc-

tion is in accord with the previous treatment of restricted relativity problems by the author, whereby the change from one observer to another is accomplished by a rotation of the hyperbolic plane $XRcT$



about R, by a hyperbolic angle u, defined as the relative rapidity of the two observers (Ref. 1, p. 154). The S′ observer's axes are RX' and RcT', and the rotation takes place about a coördinate plane YRZ. In a three-dimensional space a rotation is possible only about a point

or about a straight line, but in a four-dimensional space a rotation is also possible about a stationary plane. Thus the coördinates y and z are the same for both observers and we have

$$y = y'; \; z = z' \tag{3}$$

With four axes of coördinates, X, Y, Z, T, there are six coördinate planes, XY, YZ, ZX, TX, TY, and TZ. There are also four three-dimensional coördinate subspaces, XYZ, YZT, ZTX, and TXY. Any combination of two axes gives a coördinate plane, and any combination of three axes gives a three-dimensional coördinate subspace.

Let P be an arbitrarily chosen point in this space; its four coördinates being x, y, z, and ct. P represents a point in the three-dimensional electromagnetic field under consideration, at a particular instant of time t. At another instant, the same point in the field itself will be represented by another point, say P_1, in the representative space.

A small finite plane area K passing through P is assumed to be located in the whole four-dimensional space, rather than in one of its coördinate subspaces. In other words, none of these four coördinates is equal to zero for all the points of this plane. It will be shown later that the position and magnitude of K fully determine the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ at P. Because the area K determines the field, it will be referred to as *the field area at P*. For the present, the diagrams and the field area K need not have any physical significance.

The area K may be resolved into two component plane areas, one denoted by C and located in the coördinate subspace XYZ and the other denoted by D and parallel to the axis RcT. To indicate that the D-plane is parallel to the T-axis, the lines fs and dq are drawn parallel to this axis. That the resolution of K into C and D is unique, may be seen from the following considerations: The area K has six projections, one upon each of the six above-mentioned coördinate planes. The projections upon the coördinate planes XY, YZ, and ZX may be re-combined into a plane area in the XYZ-subspace. This area is the component C of area K. The projections of K upon the remaining three coördinate planes, TX, TY, TZ, may also be re-combined into an area D. The plane of D passes through fs parallel to the axis RcT; such an area has no projections upon the three other coördinate planes, being normal to them. Thus, both C and D have three projections each, and their sum, K, has all the six projections. Symbolically, we may write

$$K = C + D \tag{4}$$

a geometric addition of areas in a four-dimensional space being understood.

The vector **H**, normal to the plane C, is in the XYZ-subspace, and we shall assume the length of **H** to be numerically equal to that of area C. The plane D is assumed to be in the form of a rectangle whose length fs is equal to unity. Being parallel to the RcT axis, this length is also marked $\mathbf{k}_4$, the four unit vectors along the four coordinate axes being denoted by $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and $\mathbf{k}_4$, respectively. The other side of the rectangle D, denoted by fd is also marked **E** and is numerically equal to the area D, since fs is equal to unity.

In what follows we shall also need a linear vector in four dimensions Such a vector, drawn from point P, is denoted in our figure by **r** This vector is shown resolved into two components, viz., **p**, in the XYZ-subspace, and $\mathbf{p}_t$ parallel to the RcT-axis, and therefore located in the fourth dimension. In other words

$$\mathbf{r} = \mathbf{p} + \mathbf{p}_t \tag{4a}$$

a geometric addition in four dimensions being understood.

4. The Four-Curl of a Plane Vector and the Invariance of the First Two Maxwell Equations

The curl operator in a three-dimensional Euclidean space in orthogonal coördinates is defined as the operator

$$\nabla = \mathbf{k}_1 \frac{\partial}{\partial x} + \mathbf{k}_2 \frac{\partial}{\partial y} + \mathbf{k}_3 \frac{\partial}{\partial z} \tag{5}$$

This operator acts upon a linear vector **F** as though a cross product of ∇ and **F** were formed. In other words,

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$$

the components of this expression being formed according to the usual rules of outer or vector product. Thus, in equation (5) ∇ is considered to be a symbolic or operational vector, and the expression for the components of curl **F** may be written directly from the well-known expression for the components of a vector product of two linear vectors.

We shall now extend the above definition of the curl in two respects: (a) the del operator will be defined in a four-dimensional semi-hyperbolic space as four-del

$$\square = \nabla - \mathbf{k}_4 \;{}_{\partial(ct)} \tag{6}$$

where $\mathbf{k}_4$ is a unit vector in the direction of the cT-axis, and the minus sign between the x-term ∇, and the t-term is used on account of the hyperbolic properties of the XRT-plane; (*b*) the four-del operator will be applied to a plane located in our four-dimensional representative space, rather than to a linear vector.

We shall first define the vector product $\mathbf{r} \times K$ of a linear vector $\mathbf{r}$ and the plane vector K, and then deduce an expression for the four-curl K by replacing the actual vector $\mathbf{r}$ by the operational vector four-del.

The vector product of a linear vector $\mathbf{r}$ by a plane vector K will be here defined as the volume of the prism or the cylinder whose base is K and whose generators are equal and parallel to $\mathbf{r}$. In a three-dimensional space such a volume is a scalar quantity, but in a four-dimensional space it may also be thought of as a vector numerically equal to the above volume and extending into the fourth dimension, that is, normal to the subspace occupied by the prism itself. This latter vector will be referred to as the *product indicator*.

This procedure is analogous to that employed with the usual outer product of two linear vectors, say $\mathbf{p}$ and $\mathbf{g}$. In a two-dimensional space, this product can be thought of only as the scalar area of the parallelogram formed by the two vectors, but in a three-dimensional space $\mathbf{p} \times \mathbf{g}$ is interpreted as a linear vector numerically equal to the area of the above parallelogram and normal to the $\mathbf{pg}$-plane. This vector may be called the product indicator and the vector equal to the volume of the prism, $\mathbf{r} \times K$, and extending into the fourth dimension is but a natural generalization of this concept.

We have previously resolved the area K into the component C in the XYZ-subspace and the component D parallel to the T-axis (eq. 4). Similarly, the linear vector $\mathbf{r}$ has been resolved into a vector $\mathbf{p}$ in the XYZ-subspace and the component $\mathbf{p}_t$ parallel to the T-axis, as shown in Fig. 1 (eq. 4a). Consequently,

$$\mathbf{r} \times K = (\mathbf{p} + \mathbf{p}_t) \times (C + D) =$$
$$\mathbf{p} \times C + \mathbf{p}_t \times C + \mathbf{p} \times D + \mathbf{p}_t \times D \tag{7}$$

Each term on the right-hand side of this expression will be considered separately. Since both D and $\mathbf{p}_t$ are parallel to the T-axis, the volume of the prism built on these two quantities, that is, the outer product $\mathbf{p}_t \times D = 0$. In the term $\mathbf{p} \times C$, the area C may be replaced by its

positive normal, **H**, shown in the figure. The length of this normal, denoted by **H**, having been taken to be numerically equal to the area C, so that

$$(\mathbf{p} \times C)_4 = (\mathbf{p} \cdot \mathbf{H})_4 \tag{8}$$

The subscript 4 refers to the direction of the product indicator. Namely, both **p** and C lie in the XYZ-subspace, and so does the volume $\mathbf{p} \times C$ formed by them. Hence the product indicator, being normal to this subspace, is parallel to the fourth dimension, that is, to the T-axis. This is shown by the subscript 4.

In the product $\mathbf{p}_t \times C$, the cross multiplication sign is superfluous, because $\mathbf{p}_t$ is normal to the subspace XYZ, and hence is normal to any plane in it. The product indicator must be normal to both C and $\mathbf{p}_t$, and hence is in the direction of the normal **H**. Let the positive direction of **H** be defined by the equation

$$\mathbf{H} = C \times \mathbf{k}_4 \tag{9}$$

Then,

$$\mathbf{p}_t \times C = p_t\mathbf{k}_4 \times C = -(p_t\mathbf{H})_3 \tag{10}$$

a distinction being made between the vector $\mathbf{p}_t$ and its scalar value, p_t. The subscript 3 refers to the fact that the product indicator of the volume $\mathbf{p}_t \times C$ lies in the three-dimensional subspace XYZ.

As to the product $\mathbf{p} \times D$, it will be seen from Figure 1 that the plane formed by **p** and **E**, being normal to $\mathbf{k}_4$, is normal to the plane D. Consequently,

$$\mathbf{p} \times D = (\mathbf{p} \times \mathbf{E})_3 \tag{11}$$

The subscript 3 again shows that the product indicator of the volume $\mathbf{p} \times D$, being normal to $\mathbf{k}_4$, lies in the subspace XYZ.

Collecting now the foregoing terms in equation (7), gives

$$\mathbf{r} \times K = (\mathbf{p} \cdot \mathbf{H})_4 - (p_t\,\mathbf{H})_3 + (\mathbf{p} \times \mathbf{E})_3 \tag{12}$$

Replacing now the vector **r** by the operator four-del, we have to put

$$\square;\ \mathbf{p} = \nabla;\ \mathbf{p}_t = -\,\mathbf{k}_4 \frac{\partial}{\partial(ct)} \tag{13}$$

Therefore the preceding equation becomes

$$\square \times K = (\nabla \cdot \mathbf{H}_4) + \left(\frac{\partial \mathbf{H}}{\partial (ct)} + \nabla \times \mathbf{E}\right)_3 \tag{14}$$

Comparing this with eqs. (1), it will be seen that

$$\square \times K = 0 \tag{15}$$

In other words, using the language of the three-dimensional vector analysis, K is a lamellar vector. It is shown in Section 6 below that the four-del operator is an invariant to a rotation of the part of the the coördinate frame $XRcT$ by a hyperbolic angle which transforms the frame from the one for the S observer to that of the S′ observer. Consequently, the four-curl of K is also equal to zero for the latter observer, and we may write

$$(\nabla \cdot \mathbf{H}')_4 + \left(\frac{\partial \mathbf{H}'}{\partial (ct')} + \nabla \times \mathbf{E}'\right)_3 = 0 \tag{16}$$

but a sum of two terms, one of which lies in the fourth dimension and the other in the remaining subspace can be equal to zero only if each term is separately equal to zero. Thus we may write

$$\nabla \cdot \mathbf{H}' = 0; \quad \frac{\partial \mathbf{H}'}{\partial (ct')} + \nabla \times \mathbf{E}' = 0 \tag{17}$$

Comparing these equations with equations (1), it will be seen that they represent the first two Maxwell equations for the S′ observer. The relativistic invariance of these two equations has thus been proved on purely geometric grounds, assuming that the magnetic and electric force vectors at a point in our four-dimensional representative space may be identified with the vectors **H** and **E** shown in figure.

5. The Four-Divergence of a Plane Vector and the Invariance of the Last Two Maxwell Equations

Maxwell's equations (2) may be derived from the four-divergence of the plane vector K in a manner similar to that in which equations (1) are deduced above from the four-curl of K. Following the method in the preceding section, the four-divergence of K will be derived from the definition of the scalar product of a linear vector **r** and plane K in

four dimensions. In a three-dimensional space, the scalar product of two linear vectors is defined as the algebraic product of one of the vectors by the projection of the other vector upon its direction. For example, the work performed by a mechanical force, $\mathbf{F}$, over a distance, $\mathbf{s}$, is equal to $\mathbf{s}$ times the component of $\mathbf{F}$ in its direction.

We shall define the scalar product of a linear vector $\mathbf{r}$ and a plane K as the area K times the projection of $\mathbf{r}$ upon it. In this case the term "scalar" is a misnomer, and it may be more correct to refer to it as the inner product. We shall think of this product as a vector in the plane K and normal to the projection of $\mathbf{r}$ upon K. The positive and the negative directions of this product will be assumed to obey the well-known right-hand screw rule. The divergence of K will be obtained by substituting the operator four-del dot for $\mathbf{r}$ dot.

The area K has been previously resolved into C and D (Fig. 1), and the vector $\mathbf{r}$ into its components $\mathbf{p}$ and $\mathbf{p}_t$, equations (4) and (4a). Thus we may write

$$\mathbf{r}\cdot K = (\mathbf{p} + \mathbf{p}_t)\cdot(C + D) = \mathbf{p}\cdot C + \mathbf{p}_t\cdot C + \mathbf{p}\cdot D + \mathbf{p}_t\cdot D \quad (18)$$

In this expression, each of the four terms on the right-hand side will be considered separately. The term $\mathbf{p}_t\cdot C$ is equal to zero because by assumption the component plane C lies in the XYZ-subspace and $\mathbf{p}_t$ is normal to this subspace.

To evaluate the term $\mathbf{p}\cdot D$, we must first find the projection of $\mathbf{p}$ upon D. Since $\mathbf{p}$ lies in the XYZ-subspace, this projection is in the direction of $\mathbf{E}$, and the scalar product itself, by definition, being normal to this projection and lying in the plane D is in the direction of $\mathbf{k}_4$. Thus we may write

$$\mathbf{p}\cdot D = (\mathbf{p}\cdot\mathbf{E})_4 \quad (19)$$

the subscript 4 signifying that the product indicator is a vector parallel to the fourth dimension.

In the term $\mathbf{p}\cdot C$, the plane C may be replaced by its normal, $\mathbf{H}$, and the dot product converted accordingly into a cross product. Thus

$$\mathbf{p}\cdot C = (\mathbf{p}\times\mathbf{H})_3 \quad (20)$$

In accordance with the above definition of the dot product, the product indicator of $\mathbf{p}\cdot C$ lies in the plane C, which is located in the XYZ-subspace; hence the subscript 3 in the above equation.

To evaluate the term $\mathbf{p}_t\cdot D$, we note that $\mathbf{p}_t$ lies in the plane of D

in the direction of $\mathbf{k}_4$, so that numerically $\mathbf{p}_t \cdot D$ is equal to $\mathbf{p}_t D$ and the product indicator, being normal to $\mathbf{k}_4$, is in the direction $\mathbf{E}$, which latter lies in the XYZ-subspace. Remembering that $\mathbf{E}$ is numerically equal to the area D, we may thus write

$$\mathbf{p}_t \cdot D = (p_t \mathbf{E})_3 \tag{21}$$

Gathering now the three foregoing terms, we have

$$\mathbf{r} \cdot K = (\mathbf{p} \cdot \mathbf{E})_4 + (\mathbf{p} \times \mathbf{H} + \mathbf{p}_t \mathbf{E})_3 \tag{22}$$

Replacing now as before the linear vector r by the four-del operator and putting its components equal respectively to ∇ and to $-\partial/\partial ct$, the preceding expression finally becomes

$$\square \cdot K = (\nabla \cdot \mathbf{E})_4 + (\nabla \times \mathbf{H} - \partial \mathbf{E}/\partial ct)_3 \tag{23}$$

Comparing this expression with equations (2), it will be seen that the right-hand side of equation (23) represents the sum of the left-hand sides of equations (2). It will also be seen that the four-divergence of K is not equal to zero because the sum of the right-hand sides of equations (2) is not equal to zero. Thus the plane vector K is not solenoidal and its divergence may be put equal to a four-dimensional linear vector $\mathbf{J}$, where

$$\mathbf{J} = \rho_4 + \mathbf{j}_3 \tag{24}$$

this is permissible because in the Gaussian system of units the physical dimensions of electric charge density and of electric current density are the same. We thus may write

$$\square \cdot K = \mathbf{J} = (\nabla \cdot \mathbf{E})_4 + (\nabla \times \mathbf{H} - \partial \mathbf{E}/\partial ct)_3 = \rho_4 + \mathbf{j}_3 \tag{25}$$

In this expression, the terms in the fourth dimension must be equal to each other separately and so must the terms in the XYZ-subspace. This will give Maxwell's equations (2) for the S observer.

It is shown in the next section that the operator four-del is invariant to the rotation of the X- and T-axes, and by assumption, the plane K characterizes the field at a given point irrespective of the chosen axes. Hence, $\square \cdot K$ and $\mathbf{J}$ are invariants in the four-dimensional representative space. Thus, for the S′ observer we may rewrite equation (25) in the form

$$\square \cdot K = \mathbf{J} = (\nabla \cdot \mathbf{E}')_4 + (\nabla \times \mathbf{H}' - \partial \mathbf{E}'/\partial ct')_3 = \rho_4' + \mathbf{j}_3' \tag{26}$$

Equating the terms in the fourth dimension and in the XYZ-sub-space, it will be found that equations (2) hold true for the S′ observer as well as for the S observer. The relative rapidity of the two observers, in other words, the hyperbolic angle u of the rotation of the XRT-plane being entirely arbitrary, we conclude that Maxwell's equations (2) remain invariant for any two observers in relative motion to each other. This completes the proof of the relativistic invariance of Maxwell's equations (1) and (2) of electromagnetic field.

Thus, apart from the details of transformations, we have the following picture: In the representative semi-hyperbolic four-dimensional space the electromagnetic field at any point may be represented by a plane vector K independent of the coördinate axes. Its four-curl is identically equal to zero throughout and its four-divergence at any point is given by a linear four-dimensional vector $\mathbf{J}$ which is also independent of the coördinate axes. Resolving the four-curl of K into its components gives the first two Maxwell equations. Resolving the four-divergence of K and equating the components to those of $\mathbf{J}$ gives the remaining two Maxwell equations. This resolution being of the same form for any arbitrary position of the X and T axes, the invariance of the Maxwell equations is thereby proved and the constancy of the velocity of light immediately follows therefrom.

6. A Proof of the Invariance of the Four-Del Operator

The foregoing conclusions are based on the invariance of the four-del operator to the rotation of the XT-plane by an arbitrary hyperbolic angle u, about the plane YZ. The four-del operator, equation (6) written out in detail, is as follows:

$$\square = \mathbf{k}_1 \frac{\partial}{\partial x} + \mathbf{k}_2 \frac{\partial}{\partial y} + \mathbf{k}_3 \frac{\partial}{\partial z} - \mathbf{k}_4 \frac{\partial}{\partial ct} \tag{27}$$

Should this operator be invariant, it should read for the S′ observer:

$$= \mathbf{k}_1' \frac{\partial}{\partial x'} + \mathbf{k}_2' \frac{\partial}{\partial y'} + \mathbf{k}_3' \frac{\partial}{\partial z'} - \mathbf{k}_4' \frac{\partial}{\partial ct'} \tag{28}$$

where $\mathbf{k}_1'$, $\mathbf{k}_2'$, $\mathbf{k}_3'$, $\mathbf{k}_4'$ are the unit vectors along the coördinate axes for the S′ observer.

Since by assumption the rotation of the XT-plane takes place about the YZ-plane, equations (3) hold true, and also $\mathbf{k}_2 = \mathbf{k}_2'$ and $\mathbf{k}_3 = \mathbf{k}_3'$.

Thus the two middle terms of the four-del operator are invariant in the very nature of the case under consideration, and it remains to

show that the algebraic sum of the x- and t-terms remains invariant. To obtain this result, it is only necessary in equation (27) to express $\mathbf{k}_1$ and $\mathbf{k}_4$ through $\mathbf{k}_1'$ and $\mathbf{k}_4'$ and to express the partial derivatives with respect to x and t in terms of those of x' and t'.

By analogy with the usual formulas of transformation of coördinates, for a hyperbolic plane the geometric relationships between the four $\mathbf{k}$'s in question are:

$$\mathbf{k}_1 = \mathbf{k}_1' \cosh u - \mathbf{k}_4' \sinh u \tag{29}$$

$$\mathbf{k}_4 = \mathbf{k}_4' \cosh u - \mathbf{k}_1' \sinh u \tag{30}$$

In restricted relativity, the fundamental relationships between the x-coördinates and time intervals of the two observers are as follows (Ref. 1, Eqs. 17):

$$x' = x \cosh u - ct \sinh u \tag{31}$$

$$ct' = ct \cosh u - x \sinh u \tag{32}$$

In accordance with the usual rule of partial differentiation with respect to changed variables, we may write:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial ct'},\frac{\partial ct'}{\partial x} \tag{33}$$

$$\frac{\partial}{\partial ct} = \frac{\partial}{\partial ct}\frac{\partial ct'}{\partial ct} + \frac{\partial}{\partial x'}\frac{\partial x'}{\partial ct} \tag{34}$$

but from equations (31) and (32),

$$\frac{\partial x'}{\partial x} = \cosh u; \quad \frac{\partial x'}{\partial ct} = -\sinh u \tag{35}$$

$$\frac{\partial ct'}{\partial ct} = \cosh u; \quad \frac{\partial ct'}{\partial x} = -\sinh u \tag{36}$$

Consequently, equations (33) and (34) become

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \cosh u - \frac{\partial}{\partial ct'} \sinh u \tag{37}$$

$$\frac{\partial}{\partial ct} = \frac{\partial}{\partial ct'} \cosh u - \frac{\partial}{\partial x'} \sinh u \tag{38}$$

Substituting now the expressions from equations (29), (30), (37) and (38) in equation (27), and omitting the y and z terms, we obtain

$$\begin{aligned}
\mathbf{k}_1 \frac{\partial}{\partial x} - \mathbf{k}_4 \frac{\partial}{\partial ct} &= (\mathbf{k}_1' \cosh u - \mathbf{k}_4' \sinh u) \left(\frac{\partial}{\partial x'} \cosh u - \frac{\partial}{\partial ct'} \sinh u\right) \\
&\quad - (\mathbf{k}_4' \cosh u - \mathbf{k}_1' \sinh u) \left(\frac{\partial}{\partial ct'} \cosh u - \frac{\partial}{\partial x'} \sinh u\right) \\
&= \mathbf{k}_1' \frac{\partial}{\partial x'} \cosh^2 u - \mathbf{k}_4' \frac{\partial}{\partial x'} \cosh u \sinh u \\
&\quad - \mathbf{k}_1' \frac{\partial}{\partial ct'} \cosh u \sinh u + \mathbf{k}_4' \frac{\partial}{\partial ct'} \sinh^2 u \\
&\quad - \mathbf{k}_4' \frac{\partial}{\partial ct'} \cosh^2 u + \mathbf{k}_1' \frac{\partial}{\partial x'} \cosh u \sinh u \\
&\quad + \mathbf{k}_4' \frac{\partial}{\partial x'} \cosh u \sinh u - \mathbf{k}_1' \frac{\partial}{\partial x'} \sinh^2 u
\end{aligned} \tag{39}$$

After reduction, and remembering that $\cosh^2 u - \sinh^2 u = 1$, we finally obtain

$$\mathbf{k}_1 \frac{\partial}{\partial x} - \mathbf{k}_4 \frac{\partial}{\partial ct} = \mathbf{k}_1' \frac{\partial}{\partial x'} - \mathbf{k}_4' \frac{\partial}{\partial ct'} \tag{40}$$

This proves that expression (27) for $\Box$ is invariant to the transformation in question, as assumed in equation (28). The proof is independent of a particular value of rapidity u, because u drops out of the final formula.

No assumption having been made as to whether the operator four-del is used as a gradient, a divergence, or a curl, the proof of invariance holds for all three. It may be mentioned in passing that a modified operator four-del may be used with the plus sign before the t-term, provided that plus signs instead of minuses are also used in equations (29) and (30). It has been previously shown that Maxwell's equations of electromagnetic field can be expressed through the divergence and curl of an area vector K, the invariance of $\Box \cdot K$ and of $\Box \times K$ for the two observers proves the invariance of the Maxwell equations to the relativity transformation. This leads to the inevitable conclusion

that the velocity of light is the same for all observers, irrespective of their relative velocity; this being one of the fundamental postulates of relativity.

In an N-dimensional Euclidean space, the invariance of the N-del follows directly from the fact that this operator is fundamentally equal to $(d/dn)\mathbf{n}_1$, where $\mathbf{n}_1$ is a unit normal. Since this latter expression is determined solely by the configuration of the field and not by a chosen system of coördinate axes, it follows directly that N-del is invariant to a rotation or translation of the coördinate axes. However, in the case shown in the figure, the space being non-Euclidean, it has been deemed advisable to give a direct proof of the invariance of the four-del by using the relativity formulas of the transformation of the coördinates x and t.

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THE HAMILTON POSTAGE STAMP

AN ANNOUNCEMENT BY THE IRISH MINISTER OF POSTS AND TELEGRAPHS

The Minister for Posts and Telegraphs has given notice that on the 13th instant his Department will issue a special Commemorative Stamp in honour of Sir William Rowan Hamilton, distinguished Irish mathematician and physicist (1805–1865). The occasion is the centenary of the announcement by Hamilton of his discovery of Quaternions which he first disclosed in a paper read to the Royal Irish Academy on the 13th November, 1843.

The stamp which will be issued in two denominations, $^1/_2d.$ and $2^1/_2d.$, is the work of Mr. Seán O'Sullivan, R.H.A. The central feature of the design is a drawing of Hamilton from a bust by Hogan in Trinity College Library. The design combines the requirements of effective portraiture with that simplicity of treatment demanded for reproduction of postage stamp size. Light and shade are treated with an economy which does not detract from but rather helps the lifelike effect achieved.

Hamilton was born at what is now 36 Dominick Street, Dublin, on the night of 3/4 August 1805 and when less than three years of age was entrusted by his father to the care of his uncle, James Hamilton, Curate of Trim, Co. Meath, who had earned some reputation as a linguist. As a boy Hamilton was something of a prodigy and early in his life the mathematical bent of his mind became evident. In his 10th year he was matched in public with Zerah Colburn, the American calculating boy. At 12, he had studied Latin, Greek and the four leading continental languages and could also profess a knowledge of Syriac, Persian, Arabic, Sanskrit, Hindustani and even Malay. In his 17th year when reading the "Mécanique Céleste" of Laplace, he found an error in the reasoning on which one of the propositions was based. This discovery led to his introduction to Dr. Brinkley, then Astronomer Royal for Ireland.

In 1823, Hamilton became a student of Trinity College, Dublin, and in 1824 when only a second-year student read before the Royal Irish Academy a "Memoir on Caustics." Being invited to develop the subject he soon afterwards produced a celebrated paper on systems of rays and predicted "conical refraction." He proved that in certain circumstances a ray of light passing through a crystal will emerge not as a single or double ray, but as a cone of rays. His theoretical conclusions were later verified for universal acceptance by physical experiment and their verification aroused the interest of a degree comparable to the confirmation of the General Relativity Theory of Einstein by the observation of the Solar eclipse in 1919.

In 1827, while still an undergraduate, Hamilton was appointed Andrews Professor of Astronomy and Superintendent of the Observatory, and soon afterwards Astronomer Royal for Ireland.

About 1843 Hamilton began to think out clearly and to develop a new method in Mathematical analysis. On Monday, October 16, 1843, on his way to a Council meeting of the Royal Irish Academy, as he was passing Broome Bridge on the Royal Canal, the fundamental relations for which he had been in search suddenly occurred to him—the electric circuit closed in his mind, as he himself said. At the Council meeting of the Academy on that

day he announced his intention of presenting a paper on this discovery, and at the following general meeting, November 13, 1843, the first paper on Quaternions was read. The new method involved the discarding of the "Commutative" law of multiplication illustrated by the commonplace $a \cdot b = b \cdot a$.

Important, however, as was his work on Quaternions, it is for his work in Dynamics that Hamilton is really famous.

Professor Schrodinger has written in this connection: "While these discoveries (Quaternions, etc.) would suffice to secure Hamilton in the annals of both mathematics and physics a highly honourable place, such pious memorials can in his case easily be dispensed with. For Hamilton is virtually not dead, he himself is alive, so to speak, not his memory. I daresay not a day passes—and seldom an hour—without somebody, somewhere on this globe, pronouncing or reading or writing or printing Hamilton's name. That is due to his fundamental discoveries in general dynamics. The Hamiltonian principle has become the cornerstone of modern physics, the thing with which a physicist expects *every* physical phenomenon to be in conformity. When, some time ago, Einstein broached the idea of a theory 'without Hamiltonian principle,' it caused a sensation. Einstein himself was genuinely thrilled by this 'paradoxical' possibility. In point of fact, it proved to be a failure.

"The modern development of physics is continually enhancing Hamilton's name. His famous analogy between mechanics and optics virtually anticipated wave-mechanics, which did not have to add much to his ideas, only had to take them seriously—a little more seriously than he was able to take them, with the experimental knowledge of a century ago. The central conception of all modern theory in physics is 'the Hamiltonian.' If you wish to apply modern theory to any particular problem, you must start with putting the problem 'in Hamiltonian form.'

"Thus Hamilton is one of the greatest men of science the world has produced. We are proud to emphasize his Irish nationality, because as an American mathematician puts it 'One of the driving impulses behind Hamilton's incessant activity was his avowed desire to put his superb genius to such uses as would bring glory to his native land.' "

The stamps will be on sale at all post offices and will remain on sale until the end of June 1944.

The $^1/_2$ stamp is green and the $2^1/_2$ red-brown. Irish paper is being used and the printing has been done in the Stamping Branch of the Department of the Revenue Commissioners.